BUNDLE OF NILPOTENT LIE ALGEBRAS OVER A NONHOLONOMIC MANIFOLD (NILPOTENTIZATION)

A. M. Vershik and V. Ya. Gershkovich

On the tangent bundle of a nonholonomic manifold one defines a canonical structure of bundle of homogeneous nilpotent Lie algebras. On the local ring of a nonholonomic manifold one defines a quasijet structure. With the help of these constructions one uniformizes and simplifies a number of results of nonholonomic geometry.

The tangent bundle of a nonholonomic manifold is endowed with a canonical structure of bundle of homogeneous nilpotent Lie algebras. The fiber of this bundle is a nilpotent homogeneous Lie algebra which is naturally interpreted as the realization of a 0-quasijet (nonholonomic 0-jet) of the original distribution. Following this path we define quasijets of higher order.

The apparatus of quasijets lets us uniformize and simplify results relating to the geometry of nonholonomic manifolds, differential operators on them, etc. In this paper we use these constructions to get estimates of ε-balls of the Carnot-Caratheodory metric on nonholonomic manifolds, other applications will be considered separately.

We describe the constant of the paper briefly. In Paragraph 1 we recall the basic concepts relating to distributions and we introduce the category of nonholonomic manifolds; in Paragraph 2 we give a general description of homogeneous nilpotent Lie algebras; in Paragraph 3 we describe a procedure for nilpotentization and define the bundle of osculating Lie groups; in Paragraph 4 we consider sheaves of nilpotent Lie algebras for distributions in general position.

In Paragraphs 5 and 6 we define the local ring of a nonholonomic manifold and the algebra of jets of vector fields over this ring, describe their algebraic model, and define quasijets, in Paragraph 9 we give an estimate of ε-balls on a nonholonomic manifold, as a preliminary (Paragraph 8) recalling the definition of the Carnot-Caratheodory metric.

1. Nonholonomic Manifolds

We recall the necessary minimum of concepts related to distributions on manifolds. (There is a more detailed account in [1].)

By a distribution V on a smooth manifold M we shall mean a smooth subbundle \( \{ V(\mathbf{x}), \mathbf{x} \in M \} \) of the tangent bundle TM. A vector field \( \xi \) on \( M \) is called admissible if \( \xi(\mathbf{x}) \in V(\mathbf{x}) \) for any point \( \mathbf{x} \in M \). For each point \( \mathbf{x} \in M \) we define a chain of linear subspaces \( V(\mathbf{x}) = V_1(\mathbf{x}) \subset V_2(\mathbf{x}) \subset \ldots \) of the tangent space \( T_{\mathbf{x}} M \) by defining \( V_i(\mathbf{x}) \) as the linear span of the values at the point \( \mathbf{x} \) of vector fields which are Lie brackets of length \( \leq i \) of admissible vector fields.

Thus, $V_\infty = [V_i, V_i], \ldots, V_\infty = [V_i^{-1}, V_i]$. By the growth vector of the distribution $V$ at the point $x \in M$ we mean the natural numbers $n_V^i(x) \leq n_V^{i+1}(x) \leq \ldots$, where $n_V^i(x) = \dim V_i(x)$. The distribution $V$ is called regular if for each $i$ the function $n_V^i(x)$ is constant on $M$. In this case the collection of linear subspaces $\{V_i(x), x \in M\}$ forms a distribution $V_i$ on $M$. We shall call the chain of distributions $V_1 \subset V_2 \subset \ldots$ the Lie flag of the regular distribution $V$. One introduces the concepts of germ and jet of a regular distribution in the natural way; they are characterized as follows.

**Proposition 1.1.** The germ (jet) of the distribution $V$ at $0 \subset \mathbb{R}^n$ is regular if and only if $V$ is a free module over the ring of germs (jets) of smooth functions at $0 \in \mathbb{R}^n$.

The distribution $V$ is said to be completely nonholonomic if one can find a natural number $K$ such that $V_K = TM$. The smallest such $K = K_V$ is called the degree of nonholonomicity of distribution $V$. By a nonholonomic (Riemannian) manifold is meant a pair $(M, V)$ where $M$ is a smooth (Riemannian) manifold and $V$ is a regular and completely nonholonomic distribution on $M$. We shall call the pair $(n, n_i) = (\dim M, \dim V)$ the dimension of the nonholonomic manifold.

Nonholonomic manifolds $\{(M, V)\}$ form a category whose morphisms $(M, V) \rightarrow (\tilde{M}, \tilde{V})$ are smooth maps $M \rightarrow \tilde{M}$, carrying $V$ to $\tilde{V}$. A smooth manifold $M$ can be interpreted naturally as a nonholonomic manifold $(M, TM)$, i.e., endowed with the distribution $V = TM$. The categories of germs and jets of nonholonomic manifolds are introduced naturally by analogy with the classical case. The role of group of events in the category of nonholonomic manifolds is played by the nonholonomic Lie groups $(G, V)$ where $V$ is a left-invariant completely nonholonomic distribution on $G$.

Nonholonomic manifolds $\{(M, V)\}$ form a category whose morphisms $(M, V) \rightarrow (\tilde{M}, \tilde{V})$ are smooth maps $M \rightarrow \tilde{M}$, carrying $V$ to $\tilde{V}$. A smooth manifold $M$ can be interpreted naturally as a nonholonomic manifold $(M, TM)$, i.e., endowed with the distribution $V = TM$. The categories of germs and jets of nonholonomic manifolds are introduced naturally by analogy with the classical case. The role of group of events in the category of nonholonomic manifolds is played by the nonholonomic Lie groups $(G, V)$ where $V$ is a left-invariant completely nonholonomic distribution on $G$. Such a distribution is defined by a linear subspace $\mathcal{V}$ of the Lie algebra $\mathfrak{g}$ (generated by $\mathcal{V}$). We shall call the pair $(\mathcal{V}, \mathfrak{g})$ the nonholonomic Lie algebra. By a morphism of nonholonomic Lie algebras $(\mathcal{V}, \mathfrak{g}) \rightarrow (\tilde{\mathcal{V}}, \tilde{\mathfrak{g}})$ we mean a homomorphism of Lie algebras $\mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ carrying $\mathcal{V}$ into $\tilde{\mathcal{V}}$. The classical Lie theorem establishes an isomorphism of the category of nonholonomic Lie algebras and the category of germs of nonholonomic Lie groups.

It turns out that the tangent bundle of a nonholonomic manifold $(M, V)$ is endowed with a canonical structure as bundle of homogeneous nilpotent Lie algebras $\mathcal{g} \vee M$. We start with the general concept of homogeneous nilpotent Lie algebras postponing the definition of the bundle $\mathcal{g} \vee M$ to Paragraph 3.

### 2. Homogeneous Nilpotent Lie Algebras

We call a nilpotent Lie algebra homogeneous if it is isomorphic to a quotient algebra of a free Lie algebra by a homogeneous ideal of relations. A homogeneous nilpotent Lie algebra has a natural filtration $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \ldots$ where $\mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}]$. Fixing a linear space $\mathcal{V}$ in $\mathfrak{g}$ supplementary to $\mathfrak{g}_2$ defines a filtration $\mathfrak{g} = \mathcal{V} \oplus \mathfrak{g}_2$ in $\mathfrak{g}$ for which $\mathfrak{g}_2 = \mathcal{V} \mathfrak{g}_2 \mathcal{V}$.

**Proposition 2.1.** Let $\mathfrak{g}$ be a homogeneous nilpotent Lie algebra, $\text{Aut} \mathfrak{g}$ be the group of automorphisms of $\mathfrak{g}$. Then the following assertions hold: 1) $\text{Aut} \mathfrak{g}$ acts transitively on the set of subspaces $\mathcal{V}$, supplementary to $\mathfrak{g}_2$. 2) The exact sequence

$$0 \rightarrow [\mathfrak{g}, \mathfrak{g}] \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \rightarrow 0$$

defines a decomposition of the group $\text{Aut} \mathfrak{g}$ into a semidirect product:

$$\text{Aut} \mathfrak{g} \simeq \text{GL}_{n_1} \times \mathbb{R}^{n_2(n - n_1)}$$