RESONANCE PROCESSES IN MAGNETIC TRAPS

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An analysis is made of resonances between the Larmor rotation of charged particles in a magnetic field and the slow oscillations along the field lines. It is shown that under certain conditions these resonances can lead to the complete exchange of energy between the degrees of freedom of a particle and can cause its escape from the trap. Consideration is also given to the effect of resonances on adiabatic processes associated with a variation of the magnetic field as a function of time.

One of the methods of thermally isolating a plasma in order to realize a controlled thermonuclear reaction is the application of so-called adiabatic traps (traps with magnetic mirrors) proposed by G. I. Budker [1]. Similar systems have been proposed by York [2] and have been considered in [3]. In recent years this approach has received a good deal of attention [4, 5]; for this reason further investigations of systems of this kind are of great interest.

The action of an adiabatic trap is based [1] on the conservation of orbital magnetic momentum of a charged particle in a magnetic field \( I = (M v_2^{ \perp} L) \); \( v_\perp \) is the component of the particle velocity in the direction perpendicular to the magnetic field \( \mathbf{B} \). Obviously, a necessary but not sufficient condition for the realization of a trap is the possibility of containment of an individual charged particle. The lifetime of such a particle in a trap, is not infinite since the magnetic moment is only an adiabatic invariant, i.e., it can change slowly and this change leads to a redistribution of energy between the longitudinal and transverse degrees of freedom of the particle, and ultimately to its escape from the trap.

The change of an adiabatic invariant has been considered by a number of authors (cf., for example, [6–8]); however, it is only in [7] that calculations have been applied to the concrete result of a harmonic oscillator, in particular:

\[
\frac{\Delta I}{I} = \frac{2 \Delta (q)}{(2 \omega_0)^r} \cos \left( 2 \theta_0 - \frac{n \pi}{2} \right), \tag{1}
\]

Where \( I \) is the adiabatic invariant; \( \Delta (q) \) is the discontinuity of the \( q \)th derivative of \( \omega(\theta) \); \( \theta, \omega_0 \) are the phase and frequency of the oscillator at the discontinuity in the derivative. The chief shortcoming of the expression given above is its asymptotic character. This means that it is valid only for \( 1 / \omega T \to 0 \) (\( T \) is the characteristic time for a change in \( \omega(\theta) \)). For finite values of the adiabaticity parameter \( 1 / \omega T \), (1) is not always valid (the conditions for validity are given in the Appendix).

In the particular case in which \( \omega(\theta) \) is an analytic function, (1) gives \( \Delta I / I = 0 \). This means that when \( 1 / \omega T \to 0 \), the quantity \( \Delta I / I \) approaches zero faster than any power of the parameter \( 1 / \omega T \) (for example as \( e^{-\omega T} \)), but remains unknown. Thus, the usual asymptotic expansion in powers of a small parameter of the type \( 1 / \omega T \) is not feasible in the present case.

In the present paper we consider another approach to the problem. This approach is based on a simple physical model of the resonances between the Larmor rotation of the charged particle and the slow oscillations of the particle along the magnetic field lines. Such resonances are possible in spite of the difference in frequencies if the slow oscillations of the particle are anharmonic (i.e., contain higher harmonics of the fundamental frequency). In particular, the effect of such resonances leads to a change in the magnetic moment of the individual particle (without collisions).

1. BASIC EQUATIONS

In the present paper we do not attempt to obtain calculation formulas; main attention is concentrated on the physical processes which take place in the motion of a charged particle in a magnetic trap. For this reason we shall limit ourselves to an analysis of the simple Hamiltonian used in [6] (\( M = 1 \)):

\[
\mathcal{H} = \frac{p_x^2 + p_y^2 + \omega^2 (x)^2}{2} ; \quad p_x = \dot{x} ; \quad p_y = \dot{y} \tag{1.1}
\]

Here \( x \) and \( y \) are the coordinates along the lines of force and perpendicular to the lines of force, respectively; \( \omega \) is the Larmor frequency.

The equations of motion are

\[
\dot{y} = -\omega^2 y ; \quad \dot{x} = -\omega \frac{d\varphi}{dx} y^2. \tag{1.2}
\]

Since the oscillations along the \( x \) axis are slow (\( \Omega \ll \omega \), the solution for \( y \) is of the form

\[
y = p \cos \theta ; \quad \theta = \int \omega \, dt + \varphi, \tag{1.3}
\]

\((p \) is the Larmor radius of the particle) and

\[
\dot{x} = -\omega \frac{d\varphi}{dx} \frac{p^2}{2} \left( 1 + \cos 2\theta \right). \tag{1.4}
\]

Since \( \omega_0^2 / 2 = 1 \), where \( I \) is related to the magnetic movement by the relation \( I = c \mu / e \), we have

\[
\dot{x} + I \frac{d\varphi}{dx} = -I \frac{d\varphi}{dx} \cos 2\theta. \tag{1.4}
\]

* The role of resonances in change of an adiabatic invariant has been considered in [9].
Thus, the motion along the x axis is oscillatory with a potential energy $I\omega(x) = \mu H(x)$; on this oscillatory motion is superimposed a fast periodic perturbation characterized by frequency $\omega$. Usually this perturbation is neglected because $\omega \gg \Omega_1$; however, if the oscillations along the x axis contain high harmonics of the fundamental frequency it is possible to have a resonance between the fast periodic perturbation $I(d\omega/dx)\cos 2\theta$ and one of these harmonics.

The effect being described here can be approached from another point of view. We consider the equation $\dot{y} + \omega^2(t) y = 0$, in which the dependence of $\omega$ on time is related to the oscillations along the x axis. The period for a change of the function $\omega(t)$ is much larger than $1/\omega$; but if $\omega(t)$ contains high frequencies, up to $\omega$, one of these can cause a parametric resonance.

Since the total energy of the particle $E$ is conserved, both resonances being considered lead to a redistribution of the energy of the particle between the various degrees of freedom.

In order to investigate these resonances we make use of a method described in [10]. First we introduce the Hamiltonian $H_y$, which describes the motion along the y axis:

$$H_y = \frac{p_y^2 + \omega^2(t) y^2}{2}. \quad (1.5)$$

Then $\frac{d}{dt} H_y = \frac{\partial}{\partial y} \omega \omega = \omega \omega$. The change in the adiabatic invariant $I = H_y/\omega$ is given by

$$\frac{dI}{dt} = H_y'/\omega - H_y'\omega^2 = \omega (y' - \bar{y}',) \quad (1.6)$$

where the bars denote averages over phase which changes with frequency $\omega$. Regarding $\omega$ as a parameter, we obtain the frequency correction $\varphi(1.3)$ [10]:

$$\varphi = -\frac{\omega}{\dot{y}} \left[ \omega y \frac{\partial y}{\partial \varphi} \right]_0, \quad (1.7)$$

which coincides with (1.6). As for (1.7) we obtain for $\theta(1.9)$ of [10]:

$$\dot{\theta} = \frac{\Omega}{\omega} \left( \frac{\partial z}{\partial T} \right)_z I = I \left( \frac{\partial z}{\partial T} \right)_z \frac{\Omega}{\omega} \frac{d\omega}{dz} \cos 2\theta. \quad (1.11)$$

We may note that (1.7), (1.10), and (1.11) are exact for the Hamiltonian which has been used (1.1).

2. FIRST-ORDER RESONANCES

We now integrate (1.10), expanding the right side in a Fourier series. The function $2\theta$ describes the frequency-modulated oscillation:

$$\dot{\theta} = \varphi + \varphi + \sum_n \omega_n \cos 2\theta, \quad \varphi = \varphi + \varphi + \sum_n \omega_n \cos 2\theta. \quad (2.1)$$

The bars here and below denote averages over phase which change with frequency $\Omega$. Making some simple transformations we find

$$\cos 2\theta = \frac{1}{2} \sum_n (F_{1n} + F_{2n}) \cos 2(\omega t + \varphi \pm \theta), \quad (2.2)$$

where the summation is taken over both signs (both signs change simultaneously). The expansion of $\omega(t)$ has only cosine symmetry with respect to the turning point ($x=0$); the factor of 2 in (2.1) is due to the symmetry of $\omega(x)$ with respect to the median plane of the magnetic field. The Fourier coefficients $F_{1n}$ and $F_{2n}$ are determined by the relations

$$\cos \left( \sum \frac{\omega_n}{\sqrt{n}} \sin 2n\theta \right) = \sum F_{1n} \cos 2n\theta, \quad \sin \left( \sum \frac{\omega_n}{\sqrt{n}} \sin 2n\theta \right) = \sum F_{2n} \sin 2n\theta. \quad (2.3)$$

Suppose now that

$$\varphi = \sum \varphi_m \sin 2m\theta. \quad (2.4)$$

Multiplying (2.2) and (2.4) we obtain the equation

$$\frac{dI}{dt} = I \sum_{m \pm \Omega} P_i \cos \varphi_i. \quad (2.5)$$

The condition $\bar{\omega} \approx \Omega$ shows that in all the summations it is necessary to keep only one term, the frequency of change of which is close to zero. This is the resonance term, which gives the largest contribution to the change in $I$.

$$\varphi_i = 2(\omega_t + \varphi_t - \theta); \quad \varphi_i = \varphi - \pi/4, \quad (2.6)$$

$$4P_i = -\sum_{m \neq 0} \varphi_{1m} (F_{1m} + F_{2m}) + \sum_{n \neq m} \varphi_{1m} (F_{1n} + F_{2n}) \quad (2.7)$$

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