1. Introduction

In a recent paper, Lyne (1970) has looked theoretically at the problem of flow of a purely viscous liquid in a curved pipe under the influence of a periodic pressure gradient. Lyne assumes throughout that a (frequency) parameter \( \beta = \left( \frac{2 \nu}{n a^2} \right)^{1/2} \) is small, \( \nu \) being the kinematic viscosity and \( a \) the radius of the pipe section, the smallness of \( \beta \) implying that, for flow down the pipe, the viscous effects are confined to a thin layer adjacent to the wall. A solution is developed in terms of two matched asymptotic expansions, one expansion being valid near the wall and the other in the interior, away from the wall. The most striking feature that emerges from this work of Lyne's is that for sufficiently large values of the frequency \( n \) of the pressure gradient there is a region, away from the wall, in which the secondary flow in the plane of the cross-section is in the opposite sense to that for a steady pressure gradient, i.e., a region in which the liquid exhibits a kind of negative 'centrifuging effect'.

In a more recent paper, Zalosh and Nelson (1973) have looked anew at this problem and have presented a theory that allows for both high and low values of the frequency, but is restricted to regimes for which a (modified) Dean number \( L = \left( \frac{W a^2}{\nu} \right) \) is small, where \( W \) is a typical velocity along the pipe. A solution is developed by Zalosh and Nelson in infinite series form with \( L \) as expansion parameter and computations are continued as far as terms of order \( L \), the solution requiring the inversion of finite Hankel transforms, which cannot be accomplished analytically. It is shown that, for sufficiently large values of \( n \), secondary flow reversal occurs at the centre of the pipe, supporting the findings of Lyne, but no remark is made about the extent of the region of flow reversal, and of the possibility of it extending to the wall.

It is the purpose of the present paper to extend the analysis of Zalosh and Nelson to cover a wider class of liquid, namely elastico-viscous liquid, and, in particular, to ascertain the effect of elasticity on the onset of secondary flow reversal. It is remarked that

2. Mathematical formulation

We consider the class of (incompressible) elastico-viscous liquid with equations of state

\[ p^{ik} = p'^{ik} - p'' g^{ik}, \]  
\[ p'^{ik} + \lambda_1 \frac{\partial p'^{ik}}{\partial t} = 2 \eta_0 \left[ e^{(1)ik} + \lambda_2 \frac{\partial e^{(1)ik}}{\partial t} \right], \]

where \( p^{ik} \) is the stress tensor, \( p' \) an arbitrary isotropic pressure, \( g^{ik} \) the metric tensor of a fixed coordinate system \( x^i \) and \( e^{(1)ik} \) the rate of strain tensor defined in terms of the velocity field \( v_i \) by

\[ e^{(1)ik} = \frac{1}{2} (v_{i,k} + v_{k,i}). \]

In these equations \( \eta_0 \) is a constant having the dimensions of viscosity, \( \lambda_1, \lambda_2 \) constants having the dimensions of time and \( \partial/\partial t \) the convected time derivative introduced by Oldroyd (1950).
The problem of flow in a curved pipe is most easily formulated in terms of the orthogonal (toroidal) coordinate system \((r, \phi, \theta)\) introduced by Dean (1927, 1928), and illustrated in fig. 1. A cross section of the pipe is defined by \(\theta = \text{constant}\), and a point in this section by polar coordinates \(r, \phi\) with \(r = a\) representing the pipe wall. Referred to these coordinates the physical components of the velocity vector \(v_i(= u, v, w)\) and of the partial stress tensor \(p_{ik}\) are taken to be functions of \(r, \phi, t\), independent of \(\theta\). The equations of motion and of continuity then reduce to

\[
\begin{align*}
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \phi} - \frac{v^2}{r} - \frac{w^2 \sin \phi}{R + r \sin \phi} \right] &= - \frac{\partial p''}{\partial r}, \\
&\quad + \frac{\partial [r(R + r \sin \phi)p'_{(r\phi)}]}{r(R + r \sin \phi) \partial r} + \frac{\partial [(R + r \sin \phi)p'_{(\phi\phi)}]}{r(R + r \sin \phi) \partial \phi} - \frac{p'_{(\phi)} - p'_{(\phi\theta)} \sin \phi}{r(R + r \sin \phicomings}}} \\
\rho \left[ \frac{\partial v}{\partial t} + \frac{u}{r} \frac{\partial (rv)}{\partial r} + v \frac{\partial v}{\partial \phi} - \frac{w^2 \cos \phi}{(R + r \sin \phi)} \right] &= - \frac{1}{r} \frac{\partial p''}{\partial \phi}, \\
&\quad + \frac{\partial [(R + r \sin \phi)p'_{(\phi\phi)}]}{r(R + r \sin \phi) \partial r} - \frac{p'_{(\phi\phi)}}{(R + r \sin \phi) \partial \theta}, \\
\rho \left[ \frac{\partial w}{\partial t} + \frac{u}{r} \frac{\partial [(R + r \sin \phi)w]}{\partial r} + v \frac{\partial [(R + r \sin \phi)w]}{\partial \phi} \right] &= \frac{\partial p''}{(R + r \sin \phi) \partial \theta}, \\
&\quad + \frac{\partial [(R + r \sin \phi)^2 p'_{(\phi\phi)}]}{r(R + r \sin \phi)^2 \partial r} + \frac{\partial [(R + r \sin \phi)^2 p'_{(\phi\phi)}]}{r(R + r \sin \phi)^2 \partial \phi},
\end{align*}
\]

and

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{u \sin \phi}{(R + r \sin \phi)} + \frac{v \cos \phi}{(R + r \sin \phi)} + \frac{v}{r \partial \phi} = 0.
\]  

As in the case considered by Lyne and Zalosh & Nelson, we suppose the motion is due to a time-oscillatory gradient of stress along the pipe, and write \(^2\) 

\[
\frac{1}{R} \frac{\partial p_{(\phi\theta)}}{\partial \theta} = - \frac{1}{R} \frac{\partial p''}{\partial \theta} = P e^{i \omega t},
\]

where the amplitude \(P\) and the frequency \(n\) are real constants; or what is equivalent,

\[
p'' = p(r, \phi, t) - R \theta Pe^{i \omega t},
\]

\(^2\) The convention is adopted that real parts are to be understood whenever complex expressions are used to represent physical quantities.

Here \(\bar{W}\) is a characteristic velocity (which, in view of the solution to the straight pipe problem