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Nonsimilar incompressible laminar boundary-layer flows in micropolar fluids

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With 6 figures

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1. Introduction

Eringen (1) introduced the concept of a micropolar fluid to explain the behaviour of certain polymeric fluids, animal blood and liquid crystals (2–4). In this model, the fluid has microstructure displaying spin-inertia and it can sustain stress and body moments. Using this model, several authors (3–5) studied the flow problems through various configurations. Wilson (6) introduced the concept of boundary layer in micropolar fluids to study the steady, incompressible two-dimensional stagnation-point flow problem using momentum integral method. Recently, Nath (7) has obtained the similarity solution for the steady incompressible laminar boundary layer equations for micropolar fluids past a two-dimensional body. It may be noted that the similarity solutions are valid only in the neighbourhood of the stagnation point (or near the leading edge).

In this paper, we have studied the nonsimilar two-dimensional and axisymmetric boundary-layer flow problems for micropolar fluids (which have not been studied before) where the nonsimilarity arises either from the freestream velocity distribution or from transverse curvature or both. The partial differential equations governing the flow were first transformed to co-ordinate system with finite range by a transformation which maps an infinite range to a finite range and then solved by using implicit finite-difference scheme (8, 9). Computations were carried out for two problems, namely, cylinder in cross flow and the flow over a sphere and the results for a range of parameters are presented.

2. Basic equations

The boundary layer equations for steady incompressible micropolar fluid over a twodimensional or an axisymmetric body neglecting the effect of transverse curvature and the microscopic inertia term can be expressed as (6, 7)

\[ uu_x + vu_y = U U_x + v u_{yy} + K_1 \sigma_y, \]

\[ G_1 \sigma_{yy} - 2 \sigma - u_y = 0, \]

\[ [u(r/L)]_x + [v(r/L)]_y = 0. \]

The boundary conditions are

\[ u(x, 0) = v(x, 0) = 0, \quad u(x, \infty) \to U(x), \]

\[ \sigma(x, 0) = 0, \quad \sigma(x, \infty) \to 0, \]

where \( x \) and \( y \) are distances measured along and perpendicular to the body respectively, \( u \) and \( v \) are the velocity components along \( x \) and \( y \) directions respectively, \( U \) is the velocity of the potential flow at the edge of the boundary layer, \( \sigma \) is the microrotation component, \( r \) is the radius of revolution of an axisymmetric body, \( j = 0 \) for a two-dimensional body and \( j = 1 \) for an axisymmetric body, \( \mu \) is the coefficient of viscosity, \( S \) is the constant characteristic of the particular fluid, \( \rho \) is the density of the fluid, \( \nu = (\mu + S)/\rho \) is the apparent kinematic viscosity of the fluid, \( K_1 = S/\rho (K_1 > 0) \) is the coupling constant and \( G_1 (G_1 > 0) \) is the microrotation parameter, \( L \) is a characteristic length, and subscripts \( x \) and \( y \) denote partial derivatives with respect to \( x \) and \( y \) respectively.

Applying the transformations due to Görtler (10) and Meksin (11)
\[
\begin{align*}
\eta &= y(r/L)U(x)/(2vLU_x)\xi^{1/2}, \\
\psi(x, y) &= (2vLU_x)\xi^{1/2}f(\xi, \eta), \\
\sigma(x, y) &= U^2(x)/(2vLU_x)^{1/2}, \\
u &= (L/r)\psi_y, v = -(L/r)^2\psi_x.
\end{align*}
\]

To eqs. [1] to [4], we get

\[
\begin{align*}
f_{\eta\eta} + f f_{\eta\eta} + \beta(\xi)(1 - f^2) + K g_\eta &= 2\xi(f_\xi f_\eta - f_\xi f_{\eta\eta}), \\
G g_{\eta\eta} - \beta_1(\xi)(2g + f_{\eta\eta}) &= 0, \\
f(\xi, 0) &= f_\eta(\xi, 0) = 0, f_\eta(\xi, \infty) \to 1 \\
g(\xi, 0) &= 0, g(\xi, \infty) \to 0,
\end{align*}
\]

where

\[
\begin{align*}
\beta(\xi) &= 2\xi L(L/r)^2(U_x/U^2)U_x, \\
\beta_1(\xi) &= 4\xi(L/r)^2(U^2/U_x)^2, \\
K &= K_1/v, G = 2G_1 U/(vR).
\end{align*}
\]

Here \(\xi\) and \(\eta\) are transformed co-ordinates, \(\psi\) and \(f\) are the dimensional and dimensionless stream functions respectively, \(g\) is the dimensionless microrotation function, \(\beta\) is the pressure gradient parameter, \(\beta_1\) is the function depending on the streamwise co-ordinate, \(K\) and \(G\) are the dimensionless coupling and microrotation parameters respectively, \(U_\infty\) is the free-stream velocity, and subscripts \(\xi\) and \(\eta\) denote derivatives with respect to \(\xi\) and \(\eta\) respectively. It has been shown (7) that \(K < 1\) (0 < \(K < 1\)) and the case \(K = 1\) corresponds to the limiting (but non-physical) case of infinite \(S\).

The dimensionless skin-friction coefficient at the wall is (6, 7)

\[
c_f(Re_x)^{1/2} = 2(r/L)^2\left[x U/(2\xi L U_\infty)\right]^{1/2} f_{\eta\eta}(\xi, 0),
\]

where

\[
c_f = 2\tau_w/\rho U^2.
\]

Here \(c_f\) is the surface skin-friction coefficient, \(Re_x = U x/v\) is the local Reynolds number and \(\tau_w\) is the wall shear stress.

3. Transformations to finite co-ordinates

We transform eqs. [8] and [9] to a new system of coordinates wherein the infinite range of integration (0, \(\infty\)) on \(\eta\) is replaced by a finite range (0, 1) as it is more convenient from the computational view point (8). The transformation is represented by (8)

\[
\tilde{\eta} = 1 - \exp(-\alpha \eta),
\]

where the scaling factor \(\alpha\) provides an optimum distribution of nodal points across the boundary layer. We define \(f_\eta = F\), and \(\alpha(1 - \tilde{\eta}) = z\) and change the variable \(\eta\) to \(\tilde{\eta}\). Consequently, eqs. [8] and [9] reduce to

\[
\begin{align*}
z^2 F_{\eta\eta} - z(f - x)F_{\eta\eta} + \beta(\xi)(1 - F^2) + K z g_\eta &= 2\xi(F F_\xi - zF_\eta f_\xi), \\
z^2 g_{\eta\eta} - \alpha G z g_\eta - \beta_1(\xi)(2g + z F_\eta) &= 0.
\end{align*}
\]

The boundary conditions [10] are now transformed to

\[
F(\xi, 0) = g(\xi, 0) = 0, F(\xi, \infty) \to 1, g(\xi, \infty) \to 0.
\]

4. Method of solution

The solution of the partial differential eqs. [16] and [17] under conditions [18], which govern the nonsimilar flow past an arbitrary two-dimensional or an axisymmetric body, can be obtained provided \(\beta(\xi)\) and \(\beta_1(\xi)\) (which depend on the shape of the body) at every streamwise location are known. Eqs. [19] and [20] may be converted into a set of implicit finite-difference equations and the resulting linear algebraic equations which are of the tri-diagonal form can be solved by Thomas' algorithm. Since the method of solution is similar to that employed by Marvin and Sheaffer (8) and Vimala and Nath (9), who describe the method in full detail, its description is omitted here. In particular, we have solved the above equations for a cylinder and a sphere and the solution is valid up to the point of separation.

5. Cylinder in cross flow

We consider the case of flow over a circular cylinder where the nonsimilarity is due to the external velocity distribution at the edge of the boundary layer. This velocity is given by the potential theory, i.e.,

\[
U/U_\infty = 2 \sin \tilde{x}; \tilde{x} = x/R,
\]

where \(R\) is the radius of the cylinder and \(\tilde{x}\) is dimensionless distance. With \(R\) as the reference