Let us examine a two-stage circular plate, which is heated over the plane surface \( r = R_2 \) by an external medium having a temperature \( t_0 \), and where there is heat exchange with an external medium having a temperature \( t_m \) across lateral surfaces \( z = \pm h_1 \) and \( z = \pm h_2 \) in accordance with Newton's law. The section of the plate surface \( r = R_1 \) bathed by the external medium is assumed to be thermally insulated.

A half-thickness of this plate can be represented as a single entity in the form

\[
h(r) = h_1 + (h_2 - h_1) S_-(r-R_1),
\]

where \( h_1 \) and \( h_2 \) are half-thicknesses of plates \( r < R_1 \) and \( R_1 \leq r < R_2 \), and \( S_-(r-R_1) = \begin{cases} 1, & r \geq R_1 \\ 0, & 0 < r < R_1 \end{cases} \) is an asymmetric unit function.

Substituting relationship (1) in the thermal-conductivity equation for circular plates of variable thickness [1]

\[
\Delta T + \frac{h}{k} \frac{d}{dr} \left[ \frac{1}{r} \frac{dT}{dr} \right] = 0
\]

and in this case, making use of the identities established in [2], one obtains

\[
\Delta \theta - \kappa \theta - (\kappa_2 - \kappa_1) \theta S_-(r-R_1) = (1 - \varepsilon) \delta_-(r-R_1) \frac{dT}{dr} \bigg|_{r=R_1},
\]

where \( \theta = T - t_m \), \( \kappa_i = \alpha_i / h_i \) (\( i = 1, 2 \)); \( \alpha_1 \) and \( \alpha_2 \) are coefficients of heat emission from surfaces \( z = \pm h_1 \) and \( z = \pm h_2 \), respectively, \( \delta_-(r-R_1) = dS_-(r-R_1)/dr \); \( \Delta = d^2/dr^2 + 1/r \cdot d/dr \), \( \varepsilon = h_2/h_1 \); and \( \lambda \) is the thermal conductivity.

The boundary conditions can be written as follows:

\[
\theta \bigg|_{r=R_1} = t_m; \quad \frac{d\theta}{dr} \bigg|_{r=0} = 0.
\]
Fig. 1. Dependence of radial force on polar radius in circular plate of constant thickness (1) and in two-stage circular plate (2) with heat emission.

The general solution of Eq. (3) with consideration given to boundary conditions (4) can be written as [3]:

\[ \theta = \frac{\int_0^R}{\Psi(R_1, R_2)} [I_0(\kappa_1 \ell) + \chi(\ell)] S_-(r - R_1)]. \]  

(5)

Here

\[ \chi(r) = \Psi(R_1, r) - I_0(\kappa_1 \ell); \]

\[ \Psi(R_1, R_2) = q_{1K}(R_1) I_0(\kappa_2 R_2) + q_{2K}(R_1) K_0(\kappa_2 R_2), \]

where

\[ q_{1K}(R_1) = R_1 \left[ \kappa_2 V_0(\kappa_1 R_1) W_1(\kappa_2 R_2) \pm \frac{\nu_1}{\kappa_2} V_1(\kappa_1 R_1) W_0(\kappa_2 R_2) \right]. \]

The solving equation for the axisymmetric tensioning of a circular plate whose stiffness varies radially takes the form [1]

\[ \frac{d^2N_r}{dr^2} + \left( 3 \frac{r}{D_N} \frac{dD_N}{dr} \right) \frac{dN_r}{dr} + (1 - \nu) \frac{1}{D_N} \frac{dD_N}{dr} N_r + (1 - \nu^2) \frac{\alpha_t D_N}{dr} \frac{dN_r}{dr} = 0, \]  

(6)

where \( D_N = 2Eh(r)/\nu^2 \) is the plate stiffness, \( \nu \) is Poisson's ratio, \( E \) is the elastic modulus, and \( \alpha_t \) is the temperature coefficient of linear expansion.

Knowing the radial force \( N_r \), we can determine the circumferential force \( N_\psi \) from the equation [1]

\[ N_\psi = \frac{d(N_f)}{dr}. \]  

(7)

Substituting the quantity \( h(r) \) in the form (1) in Eq. (6), one obtains a differential equation with singular coefficients in the form of a Dirac \( \delta \)-function:

\[ \frac{d^2N_r}{dr^2} + \frac{3}{r} \frac{dN_r}{dr} = (e - 1) \delta_-(r - R_1) \frac{dN_r}{dr} \bigg|_{r=R_1} + (1 - \nu) \frac{e - 1}{R_1} \delta_-(r - R_1) N_r \bigg|_{r=R_1} - 2\alpha_t E \frac{1}{r} \frac{dN_r}{dr} \left[ h_1 + (h_2 - h_1) S_-(r - R_1) \right], \]  

(8)

the general solution of which, with consideration of the limitation of force \( N_\Gamma \) in the center of the plate, can be derived in the form

\[ N_r = -\frac{2\alpha_t E h_1}{\Psi(R_1, R_2)} \left( \frac{1 - \frac{R_1^2}{r^2}}{2 - \frac{R_1^2}{r^2}} \right) \left( \frac{1}{2 - \frac{R_1^2}{r^2}} \right) \left( (1 - \nu) (e - 1) - 2 \right) I_1(\kappa_1 R_1) + R_1 \kappa_1 \Psi(R_1, R_2)

+ (1 - \nu) (2 - \nu) I_1(\kappa_1 R_1) S_-(r - R_1) + \frac{1}{\kappa_1} I_1(\kappa_1 r) \left[ 1 - S_-(r - R_1) \right]. \]