A similarity treatment of the Russell-Weissenberg oscillatory experiment

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1. Introduction

The earliest known experimental investigation of viscoelastic liquids in oscillatory shearing motion is that due to Russell (4), who in conjunction with Weissenberg devised an oscillating cup arrangement in which small angular oscillations about the vertical cup axis were employed. The free surface of the liquid in the cup would then exhibit standing waves in the plane of the free surface. The shape of these waves is characteristic of the viscoelastic nature of the liquid within the cup.

Russell (4, 5) has shown a method of analysis of the standing waves made visible by a trace of aluminium powder. The waves so formed would be similar to those existing in a long cylindrical pipe full of liquid which has angular oscillation imposed on it about its axis. To assist in removing the influence of the cup base Russell floated the test fluid on a low viscosity liquid (water or mercury). This would make the correspondence between cup and angular oscillatory pipe flow even closer. The following analysis is derived with both these cases in mind.

The main lines of approach are closely related to prior publication by Harris (1, 2, 3) and co-workers concerning axial oscillatory flow within a circular pipe. The simulation of angular oscillatory flow can be undertaken along lines indicated by Harris and Jenkins (2) for axial oscillations and this will be the subject of another study. Similarly, an experimental technique exactly analogous to that used by Harris and Maheshwari (1) for axial flow, would be applicable in the present case to determine relative phase angles along the radius. Here the concern is to provide a similarly treatment analogous to that previously given by Harris for axial flow.

In contrast with axial flow, the angular flow problem has apart from the work by Russell (4, 5), received no attention.

2. Theoretical treatment

For a simple Newtonian system with polar coordinates (r, Ω), viscosity μ and fluid density ρ. Considering angular acceleration in a radial plane about the axis r = 0, the governing differential equation of motion is;

\[ \mu \frac{\partial}{\partial r} \left( r^3 \frac{\partial^2 \Omega}{\partial r \partial t} \right) = \rho r^3 \frac{\partial^2 \Omega}{\partial t^2}. \]  \[ \text{[1]} \]

The motion takes place in an infinitely long tube in which end effects are neglected and symmetry in the Ω direction is assumed.

If the motion is simple harmonic and the fluid is linear in its viscous properties (it may be up to a second-order viscoelastic fluid) a linear governing differential equation is again obtained which may be written in complex notation as,

\[ i\mu^+ \frac{d}{dr} \left( r^3 \frac{d\Omega^+}{dr} \right) = -\rho r^3 w^2 \Omega^+ \]  \[ \text{[2]} \]

where,

\[ \mu^+ = \mu' - i\mu'' \]  \[ \text{[3]} \]

and

\[ \Omega^+ = \Omega' + i\Omega'' \]  \[ \text{[4]} \]

also w is the circular frequency of motion.

Defining now the following dimensionless variables,

\[ \chi = r \left( \frac{w}{i\mu^+} \right)^{1/2} \]  \[ \text{[5]} \]

\[ \alpha = \frac{\Omega^+}{\Omega_R} \]  \[ \text{[6]} \]

In (6) Ω_R is the value of Ω^+ on r = R and is taken as the reference vector in phase space so that it is a real quantity.

Substituting [5] and [6] into [2], the similarity differential equation is obtained;

\[ \frac{d}{d\chi} \left( \chi^3 \frac{d\alpha}{d\chi} \right) + \alpha \chi^3 = 0 \]  \[ \text{[7]} \]

where α is an analytic function of χ.

Eq. [7] may be compared with [7] of the previous paper by Harris (3). Again, solutions of (7) are sought in the form of successive approximations and in series form.
2.1. Successive approximation

It is convenient to work with a coordinate system moving with the solid boundary in this case. Therefore the substitution

\[ \alpha = 1 - \eta \]

is made in eq. [7] which yields,

\[ - \frac{d}{d\chi} \left( \chi^3 \frac{d\eta}{d\chi} \right) + (1 - \eta) \chi^3 = 0. \]  \[9\]

Just adjacent to the solid boundary, \((\chi \to \chi_R)\) the inertia forces become vanishingly small compared with surface forces. In this region \(\eta \ll 1\) and \([9]\) degenerates to,

\[ \frac{d}{d\chi} \left( \chi^3 \frac{d\eta}{d\chi} \right) - \chi^3 = 0. \]  \[10\]

Eq. \([10]\) may be integrated to give,

\[ \eta = \frac{\chi^2}{8} - \frac{c_1}{2\chi^2} + c_2 \]  \[11\]

where \(c_1\) and \(c_2\) are (complex) integration constants.

The boundary conditions are

\[ \eta = 0 \quad \text{on} \quad \chi = \chi_R \]
\[ \eta \to \eta_0 \quad \text{as} \quad \chi \to \chi_R. \]

Using the first of these boundary conditions in \([11]\) gives the first approximation to \(\eta\):

\[ \eta = \frac{1}{8} (\chi^2 - \chi_R^2) - \frac{c_1}{2} \left( \frac{1}{\chi^2} - \frac{1}{\chi_R^2} \right). \]  \[12\]

The secondary boundary condition cannot be applied since the solution is not valid in the region \(\chi \to 0\).

Rewriting \([9]\) as

\[ \frac{d}{d\chi} \left( \chi^3 \frac{d\eta}{d\chi} \right) = \chi^3 (1 - \eta). \]  \[13\]

This equation now includes inertia effects and by substituting \([12]\) into \([13]\) and performing the integration the second approximation is found to be

\[ \eta = \frac{\chi^2}{8} - \frac{1}{64} \left( \frac{\chi^4}{3} - \chi^2 \chi_R^2 \right) + \frac{c_1}{4} \left( \ln \chi + \frac{\chi^2}{\chi_R^2} \right) - \frac{c_3}{2\chi^2} + c_4 \]  \[14\]

where \(c_3\) and \(c_4\) are further (complex) constants.

Applying again the first of the boundary conditions yields

\[ \eta = \frac{1}{8} (\chi^2 - \chi_R^2) - \frac{\chi^2}{64} \left( \frac{\chi^2}{3} - \chi_R^2 \right) \]
\[ + \frac{c_1}{4} \left( \ln \chi + \frac{\chi^2}{\chi_R^2} \right) + \frac{c_3}{2\chi^2} + c_4 \]  \[15\]

In eq. \([15]\) both \(c_1\) and \(c_3\) will be functions of \(\chi_R\)

\[ c_1 = c_1(\chi_R) \]
\[ c_3 = c_3(\chi_R). \]

Further approximations may be obtained by recycling the second and successive approximations in the manner described above.

2.2 Solution in series form

The similarity form \([7]\) may also provide a solution in series form in which the particular integral is,

\[ \alpha = 0 \]  \[16\]

and the complementary function is

\[ \alpha = \chi^s \sum_{n=0}^{n=n} A_n \chi^n \]  \[17\]

(compare with \([20]\) of the previous publication).

In \([17]\) the \(A_n\) are a set of complex coefficients.

Substituting \([17]\) into \([7]\) and equating coefficients of successive powers of \(\chi\) to zero yields the following series of equations;

\(s + 1\) \((s + 2) A_0 = 0\)
\((s + 2) \ (s + 1) (s + 3) A_1 = 0\)
\((s + 3) \ (s + 2) (s + 4) A_2 + A_0 = 0\)
\((s + 4) \ (s + 3) (s + 5) A_3 + A_1 = 0\)
\((s + r) \ (s + r - 1) (s + r + 1) A_{r-1} + A_{r-3} = 0.\)

The indicial equation is:

\[ s = 0 \]

and the corresponding coefficients became

\[ A_0 \neq 0 \]
\[ A_1 = A_3 = A_5 = \cdots \neq 0 \]
\[ A_2 = - \frac{A_0}{8} \]