DESIGN OF CIRCULAR RINGS

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Circular rings find many applications in engineering, both as constructional elements in their own right and also as strengthening elements of thin-walled shells subjected to localized loads.

The differential equation of the elastic line of a ring is a convenient tool with which to determine the stressed and deformation state of a ring joined to a shell and loaded by a complex system of forces and also with which to analyze rings for vibrations. In many cases allowance must be made not only for the bending of the ring but for its extensibility as well [1, 2], and also for shear of its cross section in calculations on vibrations. Neglecting the extension and shear that accompany the loading of the ring by a large number of forces leads to errors that are unacceptably large. In this connection we have derived the differential equation of the elastic line of the ring allowing for the deformations of extension and shear.

In the calculation the ring is regarded as a rod of constant cross section and small curvature, one of the principal axes of the cross section being assumed to lie in the plane of the ring. Deformations of the ring are supposed to be small. The loads are located in the plane of the ring.

Let the ring be loaded by radial and tangential loads of intensities \( q \) and \( q_\phi \), respectively, and also by a bending moment \( m \). The equations of equilibrium are found by considering an element of the ring (Fig. 1)

\[
\begin{align*}
\frac{dQ}{dq} + N - qR &= 0; \\
\frac{dN}{dq} + Q + q_\phi R &= 0; \\
\frac{dM}{dq} - QR - mR &= 0,
\end{align*}
\]

(1)

where \( Q \) and \( N \) are the transverse and normal forces; \( M \) is the bending moment.

The relationships between the force factors and the radial \( w \) and tangential \( v \) displacements of the ring allowing for extension have the form:

\[
N = \frac{EF}{R} \left( w + \frac{dv}{dq} \right);
\]

(2)
Fig. 1. Force factors acting on element of ring.

\[
M = \frac{EI}{R^4} \left( \frac{d^2w}{dq^2} - \frac{dw}{dq} \right),
\]

where \( E_1 \) and \( E_l \) denote the stiffness of the ring in extension and bending.

Eliminating the transverse force \( Q \) from equilibrium conditions (1) and utilizing Eqs. (2) and (3), we obtain the set of equations:

\[
\begin{align*}
\frac{EF}{R} \left( \frac{dw}{dq} + \frac{d^2w}{dq^2} \right) + \frac{EI}{R^3} \left( -\frac{d^2w}{dq^2} + \frac{d^3w}{dq^3} \right) - \frac{dm}{dq} - qR &= 0; \\
\frac{EF}{R} \left( \frac{dm}{dq} + \frac{d^2w}{dq^2} \right) + \frac{EI}{R^3} \left( \frac{d^2w}{dq^2} - \frac{d^3w}{dq^3} \right) + m + qR &= 0.
\end{align*}
\]

On solving system (4) we arrive at the following differential equations in the radial and tangential displacements:

\[
\begin{align*}
\frac{d^3w}{dq^3} + 2 \frac{d^2w}{dq^2} + \frac{dw}{dq} &= \frac{R^3}{EI} \left( \frac{d^2m}{dq^2} + m \right) + \frac{R^3}{EI} \left( \frac{dq}{dq} + qR - \gamma \left( \frac{d^2w}{dq^2} - \frac{dw}{dq} \right) \right); \\
\frac{d^2w}{dq^2} + 2 \frac{d^2w}{dq^2} + \frac{dw}{dq} &= -\frac{R^3}{EI} \left( \frac{d^2m}{dq^2} + m \right) - \frac{R^3}{EI} \left( \frac{dq}{dq} + qR - \gamma \left( \frac{d^2w}{dq^2} - \frac{dw}{dq} \right) \right),
\end{align*}
\]

where \( \gamma = EI/R^2EF \).

Let us consider the effect of shear on the elastic line of the ring.

We denote by \( \delta \) the angle through which the cross section is turned due to bending, and by \( \beta \) the angle due to shear. The total angle of the tangent to the elastic line of the ring will then be

\[
\theta + \beta = \frac{u}{R} - \frac{dm}{Rdq}.
\]

By approximate theory, the dependences of bending moment on angle of rotation and transverse force on angle of shear have the form:

\[
M = -\frac{EI}{R} \frac{d\theta}{dq},
\]

\[
Q = \beta GF/k,
\]

where \( GF \) is the stiffness under shear, and \( k \) is a dimensionless coefficient that depends on the shape of the cross section of the ring [3].

If we eliminate the normal force \( N \) from the equilibrium conditions (1), allow for inextensibility of the ring, and utilize equalities (7)-(9), then we obtain the following system of equations:

\[
\begin{align*}
(1 + \lambda) R \frac{d^2\theta}{dq^2} + \frac{d^2w}{dq^2} + \frac{dw}{dq} + \frac{\lambda R^2}{EI} \left[ m + R \left( \frac{dq}{dq} + qR \right) \right] &= 0; \\
-\lambda R \frac{d^2\theta}{dq^2} + R \frac{d\theta}{dq} + \frac{d^2w}{dq^2} + w - \frac{\lambda R^2}{EI} \frac{dm}{dq} &= 0,
\end{align*}
\]

where \( \lambda = kEI/GF^2 \).