APPLICATION OF A MATHEMATICAL SIMULATION METHOD REPRESENTING THE PROPAGATION OF MECHANICAL PERTURBATIONS TO THE ANALYSIS OF NONLINEAR VIBRATIONS IN BLADES SUBJECT TO COLLISIONS

B. F. Shorr and G. V. Mel'nikova

One of the most effective methods currently used for the numerical solution of complicated problems in the mechanics of deformable solids (cases involving considerable nonlinearities, variable boundary conditions, and so on) is that of the direct mathematical (computer) simulation of the propagation of the wave fronts arising as a result of the loading of the solid object. The first attempts at applying the principle to the solution of wave-propagation problems were made in [1-4] using models based on the "cross"-averaging of the parameters with respect to time and elementary lengths of the test material. In a number of cases this method led to erroneous results and instability of the solution; it was also of little use in finding the propagation velocities of the wave fronts, which were usually regarded as specified in advance. We shall consider a more precise model for treating an element of the deformable solid as part of a continuous medium [5], so ensuring correctness of the results and true stability of the solution for a wide variety of problems.

1. Let us consider the propagation of plane isothermal wave fronts in an elastic inhomogeneous rod of variable cross section \( F(x) \). Let us suppose that at an instant of time \( t_0 \) the fronts \( x_1 \) divide the rod into a series of elements of finite length \( \Delta x_i \) within each of which the stressed and inertial state of the system is uniform, i.e., the stress \( \sigma_i \), strain \( \varepsilon_i \), velocity \( v_i \), and density \( \rho_i \) are constant (Fig. 1a). The dimensions of the elements \( \Delta x_i \) and the finite time intervals \( \Delta t \) into which the process is divided will be considered as being so small that the area \( F_i \), the loads \( p_i \) referred to the boundaries of the elements, the elastic modulus \( E_i \), and other factors may be considered as step functions remaining constant in the range \( \Delta x_i \) and \( \Delta t \) and undergoing arbitrary jumps at the boundaries of the elements at instants of time \( t^* \) and \( t_0 + \Delta t \). At these instants of time, because of the change in external loads, boundary conditions, and the displacement of the wave fronts, all or some of the parameters \( \sigma_i, \varepsilon_i, v_i, \rho_i \) will change sharply by finite amounts \( \Delta \sigma_i, \Delta \varepsilon_i, \Delta v_i, \Delta \rho_i \) in an infinitely small volume at the right (+) and left (−) boundaries of element, acquiring the values

\[
\frac{\partial \Delta f^z_i}{\partial t^*} = l_i + \Delta f^z_i, 
\]

which remain constant over the time interval \( \Delta t \). The perturbations \( \Delta f^z_i \) will propagate into the \( i \)-th element with a velocity (constant over the range \( \Delta t \) and \( \Delta x_i \)) \( c_i = \Delta x_i / \Delta t \) relative to its moving boundaries (Fig. 1b). For the perturbed zones the basic physical laws lead to the following relationships.

**Law of Mass Conservation:**

\[
\rho_i^zf_i^z c_i - \rho_i^z (c_i \mp \Delta v^z_i) = 0. 
\]

The upper sign in front of the constituent factors refers (here and subsequently) to the right-hand boundary, and the lower sign to the left. For an elastic material

\[
\rho_i^z = \rho_i [1 - (1 - 2\mu_i)\Delta \varepsilon_i^z]; 
\]

\[
F_i^z = F_i [1 - 2\mu_i \Delta \varepsilon_i^z], 
\]

Fig. 1. Computing scheme: a) state at \( t = t_* \); b) at \( t > t_* \).

Fig. 2. Propagation of stress waves along a rod with inhomogeneous elements.

Fig. 3. Diagram to illustrate the calculation of paired blades: a) joining arrangement; b) boundary conditions.

where \( \mu_i \) is the Poisson coefficient of the \( i \)-th element. It follows from Eq. (2) that

\[
\Delta v^\pm_i = \pm c_i \Delta x^\pm_i. \tag{4}
\]

**Law of Momentum Conservation:**

\[
\rho_i F^+_i c_i v^+_i - \rho_i F_i (c_i \mp \Delta v^+_i) v_i = \pm (\sigma^+_i F^+_i - \sigma_i F_i), \tag{5}
\]

whence on allowing for Eqs. (3) and (4) we have

\[
\Delta \sigma^+_i = \pm \rho_i c_i \Delta u^+_i. \tag{6}
\]

After eliminating the quantity \( \Delta v^+_i \) from (4)-(6) we obtain the ordinary equation for the propagation velocity of an elastic wave

\[
c_i = \sqrt{\frac{\Delta \sigma^+_i}{\rho_i \Delta x^+_i}} = \sqrt{\frac{E_i}{\rho_i v_i}}. \tag{7}
\]

The law of mechanical-energy conservation is satisfied identically, subject to conditions (2) and (5). At the boundaries of the elements the conditions of displacement continuity should be satisfied; this is possible, subject to the initial continuity of the material, if

\[
u^+_i = \sigma^+_i \Delta t_i; \tag{8}
\]

the equilibrium conditions should also be satisfied:

\[
\sigma^+_i F^+_i = \sigma^+_{i+1} F^+_{i+1} + \rho_{i+1} F_{i+1}. \tag{9}
\]

If the relationship between the lengths of the elements is taken such as to satisfy the condition

\[
\Delta x_i = c_i \Delta t_i, \tag{10}
\]

the wave fronts in all the elements will reach the opposite boundaries at the same time, while within the bounds of each element a new homogeneous stress-inertia state will be established, with the parameters

\[
\sigma_{i+1} = \sigma^+_{i+1}; \tag{11}
\]

After writing down Eq. (6) for the \( i \)-th element and the two on either side, i.e., \((i \pm 1)\), with due allowance for the boundary conditions between them (8)-(9), we obtain the basic computing equations

\[
\Delta v^+_i = \pm \frac{\Delta \sigma^+_i}{\rho_i c_i} = \frac{\rho_{i+1} F^+_{i+1} c_{i+1} (v_{i+1} - v_i) \pm (F^+_{i+1} \sigma_{i+1} - F_i \sigma_i)}{\rho_i F^+_i + \rho_{i+1} F_{i+1} c_{i+1}}. \tag{12}
\]