AN INTEGRAL EQUATION FORMULATION OF PLATE BENDING PROBLEMS

by

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SUMMARY

The mathematical theory of thin elastic plates loaded by transverse forces leads to biharmonic boundary value problems. These may be formulated in terms of singular integral equations, which can be solved numerically to a tolerable accuracy for any shape of boundary by digital computer programs. Particular attention is devoted to clamped and simply-supported rectangular plates. Our results indicate support for the generally accepted treatment of such plates and for the intuitive picture of deflection behaviour at a corner.

1. Introduction

It has been demonstrated in a recent paper [1] that some biharmonic boundary-value problems related to two-dimensional elastostatics may be solved numerically by an adaption of Jaswon's integral equation formulation [2]. We demonstrate in the present paper that this adaption also works well for certain biharmonic problems related to the theory of thin plates loaded by transverse forces. The relevant field quantity in plate theory is the transverse deflection $w$, analogous mathematically to Airy's stress function $\chi$, and second derivatives of $w$ yield moment components just as second derivatives of $\chi$ yield stress components. An attractive feature of plate theory is that $w$ has an immediate physical significance, and of course it can be computed to a higher accuracy than moment or stress components. However, plate theory offers some special difficulties which do not arise in elastostatics. First, it is necessary to compute moments at the boundary if they have not been prescribed thereon, and this requires the differentiation of simple layer potentials at a source point on the boundary. Secondly, in the case of polygonal boundaries, complications appear at corners owing to their infinite curvatures. Finally, as regards free boundaries, higher derivative conditions enter which are not well adapted either to theoretical or numerical analysis. These difficulties make an independent treatment of plate problems necessary.

Three distinct problems are considered in this paper: clamped rectangular plates of various dimensions subject to transverse loading; the simply-supported square plate subject to uniform transverse loading; the partly clamped, partly simply-supported square plate subject to uniform transverse loading. All our results are in excellent agreement with the approximate analytic solutions quoted by Timoshenko and Woinowsky-Krieger [3]. Our results for clamped rectangular plates also agree well with those recently obtained by Morley [4] on the basis of variational principles.

As regards simply-supported rectangular plates subject to uniform transverse loading, the problem may in effect be reduced from biharmonic to harmonic function theory by omitting the boundary term $\rho^{-1} \frac{\partial w}{\partial n}$ ($\rho^{-1}$ is the
curvature and \( \frac{\partial}{\partial n} \) denotes normal derivative), an extensively employed step apparently first pointed out in the literature by Marcus [5]. This term is indeed zero along a straight line since \( \rho^{-1} = 0 \), but could possibly become finite or even infinite at a corner owing to the behaviour of \( \rho^{-1} \) there.

No systematic analytical or numerical investigation of \( \rho^{-1} \frac{\partial w}{\partial n} \) omission effects seems to be available. Our procedure here is to round off each corner by a circular arc of radius \( \rho_0 \) as described elsewhere [1], solve the complete biharmonic problem numerically retaining \( \rho_0 \frac{\partial w}{\partial n} \) along the arcs, and examine numerical behaviour as \( \rho_0 \to 0 \). We find the results for a square plate to be almost indistinguishable from those obtained by solving the reduced problem numerically. We also find that \( \rho_0^{-1} \frac{\partial w}{\partial n} \) increases, whilst \( \frac{\partial w}{\partial n} \) decreases, as \( \rho_0^{-1} \to 0 \), so implying (as will be explained later) support for Timoshenko's intuitive picture [3] of deflection behaviour near a corner [6]. A similar, though simpler, reduction occurs on omitting \( \rho^{-1} \frac{\partial w}{\partial n} \) from the boundary conditions of a simply-supported plate subject to a uniform thermal moment [7]. This problem has been treated on the same lines as the preceding, yielding similar conclusions. Further problems now under investigation are the partly clamped, partly free rectangle, and the clamped ellipse subject to a concentrated transverse load.

Three main conclusions emerge from this paper. First, the integral equation method rapidly provides a reliable overall picture of the deflection and moment distribution, though finer details are probably best supplied by more sophisticated analytical techniques such as the polar coordinate transformations of Morley [8] or of Williams [9], or the \( \lambda \)-method of Quinlan [10]. Secondly, its numerical results lend support to the widely accepted omission of \( \rho^{-1} \frac{\partial w}{\partial n} \) for rectangular plates, so enabling the analytical treatments to be correspondingly simplified. Thirdly, it appears that the corners of a simply-supported square plate behave theoretically as expected on intuitive physical grounds.

The rest of the paper divides into three main sections: thin plate theory, integral equation formulation, numerical results and comparisons.

2. Thin Plate Theory

The transverse deflection of a thin plate under a uniform load \( k \) per unit area satisfies the equation

\[
\nabla^2 (\nabla^2 w) = \nabla^4 w = k/D,
\]

where \( D \) is the flexural rigidity. With \( w \) known, the moment components at any point \( x, y \) are determined from

\[
M_{xx} = -D \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right], \quad M_{yy} = -D \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right],
\]

\[
M_{xy} = -M_{yx} = D (1-\nu) \frac{\partial^2 w}{\partial x \partial y},
\]

using the notations and conventions of Timoshenko [3]. These formulae can immediately be adapted to the boundary, \( L \), by identifying \( x = n, y = t \), where \( n, t \) denote the (inward) normal and tangential boundary variables as indicated in Fig. 1. Accordingly we write