A Theory for the Design of Thin Heat Flux Meters

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SUMMARY

Heat flux meters are analysed asymptotically for small thickness/width ratios. The gain of the meter is calculated for several examples, both two-dimensional and axisymmetric. Ideal and optimum designs for meters are suggested.

1. Introduction

Heat flux meters are devices used in micro-meteorology to measure local directional transfer of heat, e.g. vertically in soil. They usually consist of thin discs aligned with faces normal to the required direction of heat transfer. The disc contains a thermopile to measure the temperature difference across its faces.

Since such a device obviously disturbs its environment, a number of theoretical and experimental investigations have been made (e.g. Portman [5], Philip [4], and Schwerdtfeger [6]) of its calibration properties. A natural conclusion is that for minimum disturbance to the ambient heat flow, the meter should be as thin as possible. That is, if \( \varepsilon \) denotes the ratio between maximum thickness and maximum diameter, we should have \( \varepsilon \ll 1 \).

The theoretical portion of the work of Portman [5] and of Schwerdtfeger [6] is of a general dimensional and empirical nature, and is intended to provide calibration laws for arbitrary (non-small) \( \varepsilon \), even to the point of remaining valid when the “meter” has degenerated into an ideal temperature probe, a slender object with \( \varepsilon \gg 1 \). In this wide range of \( \varepsilon \), Schwerdtfeger [6] suggests that Philip’s [4] result

\[
G = \frac{K}{\kappa + (K - \kappa)H}
\]

for the gain \( G \) (ratio between measured and ambient heat flux) is of general validity*. Here \( K \) is the mean thermal conductivity of the meter material, and \( \kappa \) that of the surrounding medium. The parameter \( H \) depends only on the geometry of the meter, and Schwerdtfeger suggests the empirical law.

\[
\log H = -\frac{1}{2} \left[ \log \varepsilon + (\log^2 \varepsilon + A)^{\frac{1}{2}} \right]
\]

(1.2)

to describe its variation with thickness \( \varepsilon \). The constant \( A \) is independent of thickness and depends only on the qualitative shape of the meter sections (rectangular, elliptic etc.).

In fact (1.1) is an exact result for oblate spheroidal meters (Philip [4]), where \( H = H(\varepsilon) \) is a complicated function of \( \varepsilon \), which reduces when \( \varepsilon \) is small to

\[
H = 1 - \alpha \varepsilon
\]

(1.3)

with \( \alpha = \pi/2 \). Philip [4] suggested (1.3) as a universal law for small \( \varepsilon \), where \( \alpha \) is a shape parameter like \( A \) in (1.2), taking values near to \( \pi/2 \). However, (1.2) appears to fit experimental measurements better for large \( \varepsilon \).

* The notation used here differs considerably from that of previous authors. In particular, our \( G = \) Philip’s \( f = \) Schwerdtfeger’s \( B_n/B_o \), our \( \varepsilon = \) Philip’s \( \eta = \) Schwerdtfeger’s \( 1/G \), our \( K/\kappa = \) Philip’s \( \varepsilon = \) Schwerdtfeger’s \( k_n/k_o \), etc. Philip also uses a thickness ratio \( r \) based on square root of base area, and the parameter \( \alpha \) in equation (1.3) differs by a factor of about 0.9 from that used by Philip in conjunction with \( r \).
On the other hand, in the limit as the meter becomes infinitesimally thin ($\varepsilon \to 0$), both (1.2) and (1.3) predict that $H \to 1$, and hence (1.1) gives $G \to 1$. This is to be expected since an infinitesimally thin meter with $\varepsilon = 0$ does not disturb the environment at all. This is also true for arbitrary $\varepsilon$ when $K = K'$, as is clear physically and follows from (1.1).

Our concern in the present paper is with small but non-zero $\varepsilon$, and we show that as a first order result for small $\varepsilon$, Philip's [4] result (1.3) is indeed valid for arbitrary meter shapes. It should be noted that if $\varepsilon$ is small, (1.1) and (1.3) together reduce to

$$G = 1 + \varepsilon \left( 1 - \frac{K}{K'} \right),$$

(1.4)

an approximate formula with error $O(\varepsilon^2)$.

We provide here means of estimating the parameter $\alpha$ for general meter shapes, both for two-dimensional and axisymmetric meters. We confirm the value $\alpha = \pi/2$ for oblate spheroidal meters, but find no evidence that $\alpha$ remains close to $\pi/2$ for general shapes. Indeed zero or even negative values of $\alpha$ are possible.

The formula (1.4) has the required property that $G = 1$ when $\varepsilon = 0$ and when $K = K'$. In addition, we observe that $G = 1$ when $\alpha = 0$, to the present order of approximation with respect to $\varepsilon$. Thus thin meters with $\alpha = 0$ are "optimum", in that they appear transparent to heat. This optimum is dependent in general on the location of the thermopile within the meter; we consider in addition "ideal" meters which have $G = 1$ for every possible location of the thermopile.

The above discussion relates to meters of effectively uniform internal conductivity $K$. If $K$ varies across the diameter of the meter (as when shielding edges are present, for example), the formula (1.4) is replaced by

$$G = 1 + \varepsilon (\alpha - \beta \kappa)$$

(1.5)

where $\alpha$, as before depends only on meter shape, while $\beta$ depends on shape and on the variable meter conductivity $K$, but not on thickness $\varepsilon$ or on external conductivity $\kappa$. The most important optimisation problem now is to choose $\beta = 0$, even if we are unable to make $\alpha$ vanish, since if $\beta = 0$ we obtain a gain of $1 + \varepsilon \alpha$ which can be calibrated once and for all independent of the (usually unknown) external conductivity $\kappa$.

2. Formulation of the General Boundary Value Problem

We consider first a very general situation as sketched in Figure 1, in which the meter is an arbitrary closed body with surface $S$, situated in a thermally uniform infinite medium with a uniform temperature gradient $T^r$ at a great distance from $S$. We choose the direction of the temperature gradient as the $y$-axis, so that if $T(x, y, z)$ is the temperature of the surrounding medium,

$$T \to T^r y \text{ as } x^2 + y^2 + z^2 \to \infty.$$  

(2.1)

Figure 1. Sketch of general heat flow problem.

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