Motion of Decaying Vortex Rings with Non-Similar Vorticity Distributions*

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SUMMARY
The motion and decay of circular vortex rings with an inner viscous core is considered by systematic matching of inner and outer asymptotic expansions. The governing Navier-Stokes equations are reduced to a coupled integro-differential system. A method of construction of solutions for the integro-differential system is presented. The initial vorticity distribution may be non-similar. Also presented is a method for introducing a time shift which makes the first term in the series solution for the vorticity to be the "best" approximation. The analysis is then applied to the motion and decay of a pair of coaxial vortex rings.

1. Introduction
The motion and decay of circular vortex rings with non-similar vorticity distributions submerged in an inviscid stream is being considered. The geometry of the ring is presented in Fig. 1.

The classical inviscid theory for the motion of circular vortex rings (e.g. see [1]) contains some serious shortcomings [2]. These include: (i) the velocity at the center of the vortical core \((r = 0)\) is infinite; (ii) if the axial velocity of the ring becomes infinite as the size of the vortical core tends to zero \((\delta \to 0)\); (iii) \(\delta \neq 0\), it must then be arbitrarily assigned in order to define the velocity of the ring; and (iv) the viscous effect is ignored even though the velocity gradient in the core is large.

C. Tung and L. Ting [2] recognized that the assumption of vanishing viscous force becomes invalid near the center of the vortical core where velocities and radial derivatives are large. Therefore they divided the flow field into two regions, an inner region near the center of the vortical core where viscous forces are important, and an outer region away from the vortical core where the flow field is inviscid. Tung and Ting then introduced asymptotic expansions for

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the vorticity \( \omega \) and the stream function \( \Psi \) in both the inner viscous region and the outer inviscid region. The expansion parameter \( \varepsilon = 1/Re^2 \) was used, where the Reynolds number \( Re \) is defined as the ratio of the circulation \( \Gamma \) to the kinematic viscosity \( v \). By the technique of systematic matching [3] Tung and Ting derived a coupled system of partial differential equations in space and time and ordinary differential equations in time to govern the decay of the vorticity, the flow field and the motion of the vortex ring. In order to uncouple the partial differential equation for the decay behavior from the others, Tung and Ting assumed that the vorticity distribution is similar. This implied a restriction on the initial data. They then used this vorticity distribution to find equations for the axial and radial motion of the ring.

In the present paper no restriction is imposed on the initial vorticity distribution. The solution for the partial differential equation for the flow field and the matching condition are combined to yield an integro-differential equation in \( t \) for the axial position of the ring. The partial differential equation for the vorticity is uncoupled by the introduction of new variables in section 3. This equation is then independent of both the outer inviscid solution and the instantaneous position of the ring. A solution for the vorticity as a series of eigenfunctions is then obtained. The coefficients in the series are related to the initial data. These solutions are then combined with the remaining ordinary differential equations to form a system of integro-differential equations in \( t \) for the motion of the ring. This system is then integrated numerically for a given initial position of the ring.

Inspection of the partial differential equations for the vorticity yields that the equation is independent of the initial value of the new variables (see above), and therefore independent of a translation in these variables. A shift is chosen in section 5 which makes the second term in the series for the vorticity distribution vanish. This is equivalent to the “optimum solution” introduced by Ting and Chen [4] for boundary layer theory and subsequently used by Kleinstein and Ting [5] for heat conduction problems and by Ting [6] for two-dimensional vortices. The accuracy of one term optimum solutions as compared to the series solution is demonstrated in the study of the motion and decay of two co-axial vortex rings.

2. Equations for the Motion and Decay of the Ring

The governing equations for the leading terms for the motion and decay of circular vortex rings were first derived by Tung and Ting [2] and subsequently modified for the general three-dimensional case by Ting [6]. In this section are presented an outline of the derivation of those equations with slight modifications in order to facilitate the subsequent analyses.

The basic governing equations for the motion and decay of circular vortex rings are given by the Navier–Stokes equations. In the outer region, the flow is irrotational \( (\omega = 0) \) and the stream function \( \Psi \) is expanded in a power series in \( \varepsilon \):

\[
\Psi(t, \rho, z, \varepsilon) = \Psi^{(0)}(t, \rho, z) + \varepsilon \Psi^{(1)}(t, \rho, z) + \ldots
\]

In the inner region, the radial variable is stretched, i.e. \( \tilde{r} = r/\varepsilon \). The stream function \( \Psi \) and the vorticity \( \zeta \) in the inner region are also expanded in power series in \( \varepsilon \):

\[
\Psi(t, \tilde{r}, \theta, \varepsilon) = \tilde{\Psi}^{(0)}(t, \tilde{r}) + \varepsilon \tilde{\Psi}^{(1)}(t, \tilde{r}, \theta) + \ldots
\]

and

\[
\zeta(t, \tilde{r}, \theta, \varepsilon) = \varepsilon^{-1} \tilde{\zeta}^{(0)}(t, \tilde{r}) + \varepsilon^{-2} \tilde{\zeta}^{(1)}(t, \tilde{r}, \theta) + \ldots
\]

Substituting the above expansions into the Navier–Stokes equations, the following equations governing the two leading terms in \( \Psi \) and \( \zeta \) are obtained:

\[
\begin{align*}
\tilde{r}_{\tilde{r}}^{(0)} &= -\frac{\tilde{\Psi}_{\tilde{r}}^{(0)}}{\tilde{r}}; & \quad \tilde{r}_{\tilde{r}}^{(1)} &= -\frac{\tilde{\Psi}_{\tilde{r}}^{(1)}}{\tilde{r}}, \\
\tilde{r}_{\tilde{r}}^{(0)} - \Gamma [\tilde{r}_{\tilde{r}}^{(0)}]_r &= (\tilde{R}_0/2\tilde{R}_0)[\tilde{r}^2 \zeta^{(0)}]_r, & \quad \text{and} \\
\tilde{\Psi}_{\tilde{r}}^{(1)} r_{\tilde{r}}^{(0)} - \tilde{\Psi}_{\tilde{r}}^{(0)} r_{\tilde{r}}^{(1)} &= [\tilde{r}_{\tilde{r}}^{(0)}/\tilde{R}_0] \tilde{\Psi}_{\tilde{r}}^{(1)} \sin \theta.
\end{align*}
\]

In Eqs. (2.1) through (2.3), \( \Psi \) is the stream function of the velocity components \( \tilde{u}, \tilde{v} \) relative to