Variational Principles for Steady Heat Conduction With Mixed Boundary Conditions

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SUMMARY
For the boundary value problem of steady heat conduction with general boundary conditions a variational problem is formulated by adding a simple surface integral to Butler's volume integral.

1. Introduction

There has been considerable interest recently in formulating the problem of heat conduction as a variational problem. For example, Hays [1] gives an integral which takes on a stationary value when the temperature distribution satisfies the heat conduction equation in a region R, provided that at all points of the surface of R either the temperature is prescribed or the normal heat flux vanishes. Hays' formulation is applicable to both time-dependent and steady problems, and the conductivity and thermal capacity may be any given functions of the temperature. Butler [2] proposes a much simpler integral for steady problems with the same type of boundary conditions.

However, one often has the normal heat flux prescribed, rather than vanishing, on a portion of the bounding surface, or the even more complicated case when neither the temperature nor the normal heat flux is given, but rather a relation between them exists, as, for example, a surface heated or cooled by convection or radiation. It is the purpose of this note to show that these more complicated problems may be expressed as variational problems by adding a simple surface integral to Butler's volume integral.

Biot [3] has also given variational principles for heat flow problems and has included the types of boundary conditions considered here. However, he is primarily interested in time-dependent problems, and his integrals do not seem to reduce in the steady case to the simple forms given in the present paper.

2. Problem I: Normal Heat Flux Prescribed

Let it be required to find the steady temperature distribution, \( T(x_1, x_2, x_3) \), in a region \( R \), bounded by a surface \( S \), subject to the boundary conditions

\[ q_i n_i \text{ is prescribed on } S_2, \text{ the remainder of } S. \quad (2) \]

Here, \( q_i \) denotes the Cartesian components of the heat flux vector and \( n_i \) denotes the components of the outer unit normal to \( S \). The heat flux is related to the temperature field by the Fourier law of heat conduction,

\[ q_i = -KT_{,i}. \quad (3) \]

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where $K = K(T)$ is any given function. The conservation of energy requires that

$$q_{i,i} = 0 \text{ in } R. \tag{4}$$

Let $T^* (x_1, x_2, x_3)$ be the temperature distribution which satisfies the system (1), (2), (3), (4); and let $K^*$ and $q^*$ be the corresponding conductivity and heat flux. Further, let $T (x_1, x_2, x_3)$ be a neighboring function to $T^*$, satisfying the same boundary conditions. Then $T$ may be expressed as

$$T = T^* + \varepsilon \eta,$$  \tag{5}

where $\eta$ is an arbitrary function of the coordinates and $\varepsilon$ is a small parameter. The conductivity corresponding to the temperature field $T$ is

$$K = K^* + \varepsilon \eta K'(T^*), \tag{6}$$

where $K'(T^*)$ denotes the derivative of $K(T)$ with respect to $T$, evaluated at $T = T^*$. Similarly,

$$q_i = - [K^* + \varepsilon \eta K'(T^*)] (T^* + \varepsilon \eta, i) \tag{7}$$

which is, to the first order in $\varepsilon$,

$$q_i = q^* - \varepsilon (\eta K^*)_i.$$  \tag{7}

Since $T$ and $T^*$ must agree on $S_1$, and $q_i n_i$ and $q^*_i n_i$ agree on $S_2$,

$$\eta = 0 \text{ on } S_1; \tag{8}$$

$$q^*_i n_i = 0 \text{ on } S_2. \tag{9}$$

Define $H(T)$ to be

$$H(T) = \int K(T) dT. \tag{10}$$

Then, for $T$ near to $T^*$

$$H(T) = H^* + \varepsilon \eta K^*. \tag{11}$$

It can now be shown that the following integral assumes a stationary value when $T = T^*$:

$$I = \int_R q_i q_i dv + \int_{S_2} q_i n_i H ds. \tag{12}$$

If one expresses the right side of (12) in terms of starred functions and variations, one can then find

$$(dI/d\varepsilon)_{\varepsilon=0} = \int_R - q^*_i (\eta K^*)_i dv + \int_{S_2} n_i [ - H^*(\eta K^*)_i + q^*_i \eta K^* ] dS. \tag{13}$$

The volume integral in (13) may be written

$$-\int_R q^*_i (\eta K^*)_i dv = -\int_R (q^*_i \eta K^*)_i dv + \int_R \eta K^* q^*_i dv. \tag{14}$$

The first integral on the right side of (14) may be converted to a surface integral, and the second vanishes because $q^*_i$ must satisfy equation (4). Also, in (13), the first term in the surface integral vanishes by the boundary condition (9). Hence, (13) becomes

$$(dI/d\varepsilon)_{\varepsilon=0} = \int_S q^*_i \eta K^* n_i ds + \int_{S_2} q^*_i \eta K^* n_i ds. \tag{15}$$

But in (15), there is no contribution from the integration over $S_1$, because $\eta$ vanishes there by boundary condition (8). Hence, (13) becomes

$$(dI/d\varepsilon)_{\varepsilon=0} = 0 \tag{16}$$

which proves that $I$ is stationary when $T = T^*$.  