PONTRYAGIN'S MAXIMUM PRINCIPLE FOR A CONSTRAINED SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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SUMMARY

Recently TIMMAN [2] succeeded in setting up a theory of optimization by applying variational principles to problems of mathematical programming and control theory. These principles may be considered as basic when dealing with problems of optimization theory. In this paper we are concerned with a general problem of control theory: inequality-constraints for both the control-variables and the state-variables are taken into account. The point is to derive necessary conditions for the optimal control, which is such that the solution of a set of ordinary differential equations minimizes some given integral. Moreover end-conditions will be considered.

1. Introduction

In problems of control theory the state \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of some system is determined by the choice of a set of control functions. The problem of finding the optimal control \( u(t) = (u_1(t), u_2(t), \ldots, u_m(t)) \) that maximizes or minimizes some functional is the fundamental problem of optimal control theory. In some fashion this problem has been treated by Pontryagin [1] who derived the well known maximum principle, a set of necessary conditions for the optimal values of the control-variables. Recently Timman [2] succeeded in deriving the necessary conditions for an even more general problem by applying elementary variational methods to control problems.

In practical applications the control-variables \( u_1, u_2, \ldots, u_m \) are generally subject to a set of constraints:

\[
\varphi_j(u, t) \leq 0, \quad j = 1, 2, \ldots, r
\]

and also the state-variables \( x_1, x_2, \ldots, x_n \) are subject to a number of constraints:

\[
g_k(x, t) \leq 0, \quad k = 1, 2, \ldots, \nu.
\]

In the following we shall be concerned with the problem of finding a control \( u(t) = (u_1(t), u_2(t), \ldots, u_m(t)) \) such that the arc \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) from a fixed point \( (X_0, T_0) \), \( X_0 = x(T_0) \) to a given point \( (X_1, T_1) \), \( X_1 = x(T_1) \) in \( (x, t) \)-space, which arc satisfies a set of ordinary differential equations:

\[
\frac{dx_i}{dt} = f_i(x, u, t), \quad i = 1, 2, \ldots, n,
\]

minimizes the integral:

\[
\int_{T_0}^{T_1} F(x, u, t) dt
\]
Moreover the control-variables are subject to the constraints (1) and the state-variables are subject to the constraints (2). The variational approach as developed by Timman [2] is basic in the derivation of the necessary conditions for the optimal values of the control-variables. The formulation of the set of necessary conditions differs slightly from that given by Pontryagin [1], though both formulations are equivalent, as far as Pontryagin has been occupied with a similar problem.

2. The set of necessary conditions.

Let \( u(t) = (u_1(t), u_2(t), \ldots, u_m(t)) \) denote the optimal values of the control-variables and assume that the curve \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) from a fixed point \( (X_0, T_0) \) to some given point \( (X_1, T_1) \), \( X_1 = x(T_1) \) in \((x, t)\)-space satisfies the differential equations:

\[
\dot{x}_i = f_i(x, u, t), \quad i = 1, 2, \ldots, n, \tag{5}
\]

subject to the conditions:

\[
\varphi_j(u, t) \leq 0, \quad j = 1, 2, \ldots, r, \tag{6}
\]

\[
g_k(x, t) \leq 0, \quad k = 1, 2, \ldots, \nu \tag{7}
\]

and minimizes the integral:

\[
\int_{T_0}^{T_1} F(x, u, t) dt \tag{8}
\]

as compared with all curves joining \((X_0, T_0)\) to \((X_1, T_1)\) and satisfying the same conditions (5), (6) and (7).

The functions \( F, f_i, \varphi_j \) and \( g_k \) are supposed to be of class \( C^2 \) in \((x, u, t)\)-space. The curve \( x(t) \) is an extremal from \((X_0, T_0)\) to the point \((X_1, T_1)\). The conditions (7) define an admissible region \( R \) in \((x, t)\)-space. It is assumed that there exists a field of such extremals from \((X_0, T_0)\) to the points \((x, t)\) in some neighbourhood of \((X_1, T_1)\) in \( R \); among all curves from \((X_0, T_0)\) to a point \((x, t)\) subject to the conditions (5), (6) and (7) an extremal has the characteristic property that it minimizes the integral

\[
\int_{T_0}^{T_1} F(x, u, \tau) d\tau. \tag{9}
\]

Let \( J(x, t) \) denote this integral along an extremal from \((X_0, T_0)\) to \((x, t)\). Now consider the extremal \( x(t) \) from \((X_0, T_0)\) to \((X_1, T_1)\). It is assumed that the right half open interval \([T_0, T_1)\) is the union of a number of such disjoint right half open sub-intervals that throughout the interior of a sub-interval certain constraints in (6) and (7) vanish whereas the other constraints are less than zero*. Let \([\tau_0, \tau_1]\) be such a sub-interval and suppose that for \( \tau_0 < t < \tau_1 \) the conditions

\[
\varphi_j(u, t) = 0, \quad j = 1, 2, \ldots, q \leq r, \tag{10}
\]

\[
g_k(x, t) = 0, \quad k = 1, 2, \ldots, \mu \leq \nu, \tag{11}
\]

whereas

* One might assume that \([T_0, T_1]\) is the union of a countable sequence of such intervals.