A Remark on the Strong Law of Large Numbers

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Abstract

The strong law of large numbers for independent and identically distributed random variables \( X_i, i = 1, 2, 3, \ldots \) with finite expectation \( E|X_1| \) can be stated as, for any \( \varepsilon > 0 \), the number of integers \( n \) such that

\[
|n^{-1} \sum_{i=1}^{n} X_i - EX_1| > \varepsilon, \quad N_\varepsilon
\]

is finite a.s. It is known that \( EN_\varepsilon < \infty \) iff \( \sum_{i=1}^{\infty} EX_i < \infty \) and that \( \varepsilon^2 EN_\varepsilon \to \text{var} X_1 \) as \( \varepsilon \to 0 \), if \( \sum_{i=1}^{\infty} EX_i^2 < \infty \). Here we consider the asymptotic behaviour of \( EN_\varepsilon(n) \) as \( n \to \infty \), where \( N_\varepsilon(n) \) is the number of integers \( k \leq n \) such that

\[
|k^{-1} \sum_{i=1}^{k} X_i - EX_1| > \varepsilon \quad \text{and} \quad EX_1^2 = \infty.
\]

1. Introduction

Let \( X_i, i = 1, 2, 3, \ldots \), be a sequence of independent and identically distributed random variables with distribution function \( F(x) \). Let \( S_n = \sum_{i=1}^{n} X_i \). If \( E|X_1| < \infty \), define for \( \varepsilon > 0 \)

\[
A_n = \{ |n^{-1} S_n - EX_1| > \varepsilon \}.
\]

Then the strong law of large numbers can be formulated in the form

\[
\sum_{n=1}^{\infty} IA_n < \infty \quad \text{a.s. for all } \varepsilon > 0,
\]

where \( IA_n \) is the indicator function for the set \( A_n \). Erdös [1] showed that \( EN_\varepsilon(\infty) < \infty \) iff \( \sum_{i=1}^{\infty} EX_i^2 < \infty \). Heyde [3] was concerned with the behaviour of \( EN_\varepsilon(\infty) \) as \( \varepsilon \to 0 \): if \( \text{var} X_1 = \sigma^2 < \infty \) then \( \varepsilon^2 EN_\varepsilon(\infty) \to \sigma^2 \) as \( \varepsilon \to 0 \). This result follows also from the paper of Müller [4]. In this note we shall analyze the behavior of \( EN_\varepsilon(n) := E \sum_{k=1}^{n} IA_k \) as \( n \to \infty \), if \( \sigma^2 = \infty \). We obtain the following
Theorem: If \(1 - F(x) + F(-x) \sim c x^{-\alpha} \) for \(x \to \infty\), \(1 < \alpha \leq 2\), \(c > 0\), then

- if \(\alpha < 2\): \(\mathbb{E}N_\varepsilon(n) \sim c_\varepsilon n^{2-\alpha}\) as \(n \to \infty\) with \(c_\varepsilon = \varepsilon^{-\alpha} c / (2 - \alpha)\)
- if \(\alpha = 2\): \(\mathbb{E}N_\varepsilon(n) \sim c_\varepsilon \log n\) as \(n \to \infty\) with \(c_\varepsilon = \varepsilon^{-2} c\).

2. Proof of the Theorem

We suppose without loss of generality \(\mathbb{E}X_1 = 0\). Let \(\varepsilon > 0\). We have \(\mathbb{E}N_\varepsilon(n) = \sum_{k=1}^{n} P(|S_k| > k \varepsilon)\).

a) Upper estimation: Let \(\theta > 0\) and define for \(i, 1 \leq i \leq k\)

\[ U_i = \begin{cases} X_i & \text{if } |X_i| \leq \varepsilon k (1 - 3 \theta) \\ 0 & \text{if } |X_i| > \varepsilon k (1 - 3 \theta) \end{cases} \quad \text{and} \quad V_i = X_i - U_i. \]

So \(P(|\sum X_i| > \varepsilon k) \leq P(|\sum U_i| > \varepsilon k) + P(\sum V_i \neq 0)\).

\(P(\sum V_i \neq 0) \leq k P(|X_1| > \varepsilon k (1 - 3 \theta)) \leq c (1 - 3 \theta)^{-\alpha} \varepsilon^{-\alpha} k^{1-\alpha} + o(k^{1-\alpha}).\)

Define \(U'_i = U_i - E U_i\). We know that \(E U_i \to E X_i = 0\) as \(k \to \infty\). Therefore

\[ P(|\sum U_i| > \varepsilon k) \leq P(|\sum U'_i| + k |E U_1| > \varepsilon k) \leq P(|\sum U'_i| > \varepsilon k - o(k)) \leq P(|\sum U'_i| > \varepsilon k (1 - \theta)) \]

for \(k\) sufficiently large. By the definition of \(U'_i\) and \(U_i\) we get \(|U'_i| \leq |U_i| + |E U_i| \leq \varepsilon k (1 - 3 \theta) + o(1) \leq \varepsilon k (1 - 2 \theta)\) for \(k\) large.

We want to use the inequality (43) in Fuk and Nagaev [2], which was obtained independently by Bennett and Hoeffding. For that purpose we calculate the variance of \(\sum U'_i\):

\[ \text{var}(\sum U'_i) = k \text{var}(U'_1) \leq k E U_1^2 \leq \begin{cases} \tilde{c} k (\varepsilon k (1 - 3 \theta))^2 - \alpha & \text{if } \alpha < 2 \\ \tilde{c} k \log \varepsilon k (1 - 3 \theta) & \text{if } \alpha = 2. \end{cases} \]

Now, specializing the inequality (43) of Fuk and Nagaev, we obtain

\[ P(|\sum U'_i| > \varepsilon k (1 - \theta)) \leq 2 \exp \left\{ -\frac{1 - \theta}{1 - 2 \theta} \log k^{\alpha - 1} + o(\log k) \right\} = o(k^{1-\alpha}) \quad \text{since } \theta > 0. \]