The master two-loop diagram with masses

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Abstract. In every mass case needed for QCD and QED two-point functions, the most difficult two-loop scalar Feynman diagram is reduced, by a systematic dispersive method, to a single integral of logarithms, whose expansion is obtained for large and, when appropriate, small momenta. The new results for the case with an intermediate state comprising three massive particles are needed for the two-loop calculation of fermion propagators.

1 Introduction

Much effort has been devoted in recent years to the calculation of Feynman diagrams encountered in perturbative QCD. In the main it has been concentrated on massless diagrams, since the asymptotic freedom of the theory means that light-quark masses have only minor significance, as chiral-symmetry-breaking parameters, in the large-momentum limit where perturbation theory is applicable.

Massless multi-loop diagrams have turned out to be surprisingly amenable to calculation by essentially algebraic methods, the most notable achievement being an algorithm, for all four-loop counterterms, involving no integration or infinite summation of any kind [1]. We have given evidence [2] that this fortunate state of affairs may persist at the level of five-loop renormalization. But with the introduction of quark masses such simplicity disappears and one is back to the old situation of QED, where formidable combinations of Spence functions occur even at the level of the imaginary part of the two-loop photon propagator [3].

This paper is devoted to a systematic analysis of the finite, scalar two-loop diagram of Fig. 1 in all the mass cases needed for QCD and QED. Since spin becomes an essential complication only at the three-loop level [1], we refer to this diagram as the master two-loop diagram. Scalar products of momenta produced by fermion and vector-boson spin sums can always be cancelled against propagator terms in the denominators of the momentum-space integrands of the two-loop diagrams contributing to QCD and QED two-point functions, giving this scalar diagram and ones that are simpler by virtue of having fewer propagators and can moreover be obtained by differentiation of the master diagram with respect to masses and the external momentum.

It is this master diagram that is the main obstacle to be overcome in obtaining two-loop perturbative contributions to the current correlators used in QCD sum rules [4-10] for mesons containing at least one heavy quark. Moreover it is needed, in a more difficult mass case, to obtain two-loop fermion propagators, for which only the counterterms have so far been calculated in QCD [11]. Work is in progress [12] on the full quark propagator at the two-loop level of the minimal subtraction scheme but has been impeded by the lack of results for this mass case of the master diagram, a situation here remedied.

Here we develop a systematic dispersive method from which every physically interesting mass case of the master diagram can eventually be reduced to single integrals of elementary functions, as opposed to the fourfold integrals that result from the use of Feynman parameters.

Moreover we are able to obtain explicit expansions at large and small momenta by the use of the differential equations satisfied by these integrals. The large-momentum expansions of bosonic propagators are needed to analyze the effects of quark masses at high energy [9, 13], while the small-momentum expansions are needed to obtain the radiative corrections to the moments of the QCD spectral functions used in sum rules for charmonium [4, 6, 9] and for mesons containing one heavy quark [5, 9], where asymptotic freedom appears at zero momentum because of the large scale set by the heavy-quark mass [4].

The remainder of the paper is organized as follows.

Section 2 gives a generic dispersive method for reducing the diagram to a single integral of logarithms and, possibly, complete elliptic integrals of the third kind.

In Sect. 3 we study all the physically relevant cases involving N massive particles of a common mass and (5−N) massless ones. These are the N = 2, 3, 4 cases of Fig. 3b, c, d and the alternative N = 3 case of Fig. 3e. We use the physically uninteresting, but simplest non-trivial case of Fig. 3a, with N = 1, for illustrative purposes. It transpires that only in the N = 3 case of Fig. 3c does one encounter...
an elliptic integral, and even here it is of the first kind. One of several checks applied to our results comes from a highly non-trivial sum rule, whose origin is illustrated in Fig. 4 for the \( N = 3 \) case of Fig. 3c.

Section 4 deals with the remaining, unequal-mass case of Fig. 5, with a result which we quoted in [8], having obtained it by the use of differential operators that reduce the diagram to the much simpler one in Fig. 6. In this paper the result follows directly from the general method and is found to be equivalent to one obtained by Rufa [14], despite the much greater complexity of his expressions.

Section 5 gives a summary and discussion. The coefficients of the large- and small-momentum expansions in the \( N = 2, 3, 4 \) cases are given, through fifth order, in Table 1, which may be extended ad libitum by the use of differential equations that we obtain en route.

Wherever possible the derivations refer to standard mathematical texts [15, 16, 17] for details. Where these proved insufficient for our purposes an attempt has been made to indicate the procedure, though much analysis may be required to progress from one formula to the next. Our polylogarithms are specifically chosen to simplify the analysis and are combinations of those found in Lewin’s definitive text [15]. The list of their properties in Subsect. 3.1 is vital to progress and greatly reduces the computational and notational burdens of Sects. 3 and 4.

2 Dispersive contributions

The two-loop diagram of Fig. 1, with scalar propagators, leads to an integral which we normalize as follows:

\[
I(q^2) = - \frac{q^2}{\pi^2} \int d^4q P_1(l)P_2(l-q)P_3(l-k)P_4(k)P_5(k-q)
\]

where \( P_i(l) \equiv \sqrt{l^2 - m_i^2 + i\epsilon}^{-1} \). The function (1) is analytic in the \( q^2 \)-plane cut along the positive real axis, with the lowest branchpoint at \( s_0 = \min([m_1 + m_2]^2, [m_4 + m_5]^2) \).

It vanishes at the origin (unless all the masses vanish) and is bounded at infinity, since \( I(-\infty) = 6\zeta(3) [1] \). The discontinuity across the cut is given by

\[
\sigma(w) = - \frac{1}{\pi} \text{Im} I(w^2 + i\epsilon)
\]

\[
= \{ \Theta(w - m_1 - m_2)\sigma_a(w^2) + (1 \leftrightarrow 4, 2 \leftrightarrow 5) \} + \{ \Theta(w - m_3 - m_4)\sigma_b(w^2) + (1 \leftrightarrow 2, 4 \leftrightarrow 5) \}
\]

where \( \sigma_a, b \) correspond to the cuts of Figs. 2a, b.

The contribution \( \sigma_a \) of Fig. 2a involves a form factor that can in turn be evaluated dispersively by the cut of Fig. 2c.

![Fig. 2a-c. With the cut lines on-shell one obtains: a the dispersive contribution (2); b the dispersive contribution (3); c the discontinuity of the form factor needed in a](image)

Fig. 2c. This yields an inverse hyperbolic tangent in the discontinuity of the form factor, after integration over the momentum transfer in the scattering process \( 1 + 2 \rightarrow 4 + 5 \), resulting in

\[
\sigma_a(w^2) = - \int_{m_4 + m_5}^\infty dx \frac{4x}{x^2 - w^2} \frac{\Delta(w^2, m_1^2, m_2^2)}{\Delta(x^2, m_1^2, m_2^2)} \cdot T(x^2, m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)
\]

where the principal value is to be taken and

\[
\Delta(a, b, c) = \left\{ a^2 + b^2 + c^2 - 2(ab + bc + ca) \right\}^{1/2}
\]

\[
T(s, a, b, c, d, e) = \tanh^{-1}\left( \frac{\Delta(s, a, b, d, e)}{s^2 - (s + b - 2c + d + e) + (a - b)(d - e)} \right).
\]

To evaluate the contribution \( \sigma_a \) of Fig. 2b we must integrate over the Dalitz plot for the process \( w \rightarrow m_2 + m_3 + m_4 \). The integration over the invariant mass in the \( 2 + 3 \) channel also gives a \( \tanh^{-1} \) function, which must then be integrated over the invariant mass in the \( 3 + 4 \) channel, giving

\[
\sigma_b(w^2) = \int_{m_3 + m_4}^\infty dx \frac{4x}{x^2 - m_1^2} T(x^2, w^2, m_2^2, m_3^2, m_4^2, m_5^2).
\]

The \( \tanh^{-1} \) functions can now be removed from each of the integrands of (2, 3) by taking the derivative of \( \sigma \), which is sufficient to evaluate the diagram (1) from the dispersion relation

\[
I(q^2) = - \int_{s_0}^\infty ds' \sigma'(s') \left\{ \log\left( \frac{1 - q^2}{s'} \right) - \log\left( \frac{1 - q^2}{s_0} \right) \right\}
\]

where the second logarithm may be dropped if \( \sigma(s_0) = 0 \) (as is the case when \( s_0 = 0 \)).

The derivative of \( \sigma_a \) of (2) may be evaluated using

\[
\frac{\partial}{\partial w}\left( \frac{4x}{x^2 - w^2} \frac{\Delta(w^2, m_1^2, m_2^2)}{\Delta(x^2, m_1^2, m_2^2)} \right)
\]

which permits an integration by parts, giving an integrand involving only one square root of a quadratic and hence yielding \( \sigma_a \) as a complicated combination of \( \tanh^{-1} \) functions. By contrast the derivative of \( \sigma_b \) of (3) gives an integrand involving the square root of a quartic in the general mass case, resulting in complete elliptic integrals of the third kind. The appearance of elliptic integrals should come as no surprise, since even the area of the Dalitz plot involves them. However, when \( m_1 m_2 m_3 = m_2 m_4 m_5 = 0 \) an evaluation of \( \sigma'(s) \) in terms of \( \tanh^{-1} \) functions is possible, since in these cases each three-body cut involves at least one massless particle.