Bosonic zero modes in discretized light-cone field theory

Gary McCartor and David G. Robertson
Department of Physics, Southern Methodist University, Dallas, TX 75275, USA
Received 30 October 1991

Abstract. It is shown that for theories with bosonic fields a constrained zero mode is a necessary ingredient for a consistent discretized light-cone quantization (DLCQ). Inclusion of this zero mode is shown to remove a non-covariant, quadratically divergent contribution to the fermion self-energy in 3 + 1 dimensional Yukawa theory which would otherwise be present. It is further shown to result in a fully consistent set of Heisenberg equations. The possibility of maintaining parity in DLCQ is discussed.

Introduction

There has been considerable interest recently in the use of light-cone quantization [1] coupled with a Tamm-Dancoff truncation of the space of states [2, 3] to solve quantum field theories nonperturbatively [4–17]. These techniques have already been applied with a great deal of success to various 1 + 1 dimensional models [4–7, 9, 10, 17], and significant work on 3 + 1 dimensional QED has also been reported [14–16]. Results for the positronium spectrum and wavefunctions are qualitatively good, and the quantitative agreement is expected to improve with better treatments of e.g. gauge invariance, and streamlined numerics. These successes are stimulating further development of the method, the ultimate goal being a nonperturbative calculation of the hadronic spectrum and wavefunctions starting from microscopic QCD.

Most of the actual numerical work thus far has been within the framework of “discretized light-cone quantization” (DLCQ), first introduced in [4]. In this approach the theory is defined in a box, of length 2L in the x⁻ direction and 2L₁ in the transverse directions (see Appendix A for a discussion of our conventions). The fields are required to be periodic or antiperiodic on these intervals, which leads to a discrete set of allowed Fourier modes. This denumerably infinite set of states is then truncated by momentum cutoffs, leaving a finite set, and from here the calculations begin.*

As it is usually formulated, however, DLCQ has some puzzling features. In order to appreciate these let us consider the scalar-coupled Yukawa theory in 3 + 1 dimensions, defined by the action**

\[
S = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2 + \frac{i}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \psi) \gamma^\mu \psi) - m \bar{\psi} \psi - g \bar{\psi} \phi \psi \right].
\] (1)

The DLCQ Hamiltonian operator \( P^- \) may be constructed following [14]. Briefly, we derive from (1) the energy-momentum tensor \( T^{\mu \nu} \) in the standard fashion. \( P^- \) is then obtained by integrating \( T^{\mu \nu} \) over the surface \( x^+ = 0 \):

\[
P^- = \frac{1}{2} \int d^2x d^2T^{+-}.
\] (2)

\[
T^{+-} = (\partial_\phi)^2 + \mu^2 \phi^2 - i \bar{\psi}_-(\partial_+ \psi_-) + h.c.
\]

\[
+ 2 \psi^+ \left[ -i x^0 \partial^0 + m \bar{\beta} + g \beta \phi \right] \psi_+ + h.c.,
\] (3)

where \( \alpha^i = \gamma^0 \gamma^i \) and \( \beta = \gamma^0 \) are the original Dirac matrices. The field \( \psi_- \) is nondynamical and must be eliminated via the constraint relation

\[
i \bar{\psi}_- \psi_- = \frac{1}{2} \left[ -i x^0 \partial^0 + m \bar{\beta} + g \beta \phi \right] \psi_+.
\] (4)

The dynamical fields \( \psi_+ \) and \( \phi \) are expanded in plane waves on \( x^+ = 0 \) and required to satisfy certain boundary conditions. We shall here take \( \psi_+ \) to be antiperiodic in \( x^- \) and periodic in \( x \), so that

\[
\psi_+(x) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \chi_{\mathbf{k}} [b_{\mathbf{k} \mathbf{x}} e^{ik \cdot x} + d^\dagger_{\mathbf{k} \mathbf{x}} e^{ik \cdot x}]
\] (5)

where \( \Omega = 8L L^2 \) is the “spatial” volume, and the sums run over all \( \mathbf{k} \).

* Other workers have recently advocated the use of e.g. Gaussian basis functions, instead of plane waves [18, 17]. We shall not address these alternative formulations directly in this paper.

** For simplicity we have chosen to exclude a possible \( i \phi \) term in (1.1). This has no effect on the specific calculations we present here.
over the allowed momenta
\[ k^+ = \frac{n\pi}{L}, \quad n = 1, 3, 5, \ldots; \quad k^+ = \frac{n'\pi}{L_\perp}, \quad n' = 0, \pm 1, \pm 2, \ldots \]

The scalar field is taken to be periodic in both \( x^- \) and \( x \). It is convenient to explicitly separate off the \( x^- \)-zero mode; we write
\[ \phi(x) = \phi_0(x) + \tilde{\phi}(x), \]
with \( \phi_0 \) independent of \( x^- \) and
\[ \tilde{\phi}(x) = \frac{1}{\sqrt{\Omega}} \sum \frac{1}{\sqrt{q^+}} \left[ a_q e^{-ik^+x} + a_q^* e^{ik^+x} \right]. \]

In (8) \( q \) takes on the values
\[ q^+ = \frac{m\pi}{L}, \quad m = 2, 4, 6, \ldots; \]
\[ q^+ = \frac{m'\pi}{L_\perp}, \quad m' = 0, \pm 1, \pm 2, \ldots \]

In the usual formulation of DLCQ one sets \( \phi_0 = 0 \) [14]. This is motivated by the observation that any zero mode in \( x^- \) has vanishing conjugate momentum, since \( \Pi_{\phi} \sim \partial_- \phi \), and so is not a true dynamical degree of freedom.

The canonical commutation relations to be imposed are [19, 20]
\[ \{\psi_+, (x^+, x), \psi_+, (x^+, x')\} = (A_+)_{q\bar{q}} \delta^{(3)}(x-x') \]
and
\[ \{\tilde{\phi}(x^+, x), \partial_- \tilde{\phi}(x^+, x')\} = i\delta^{(3)}(x-x') - \frac{i}{2L}\delta^{(2)}(x-x'). \]

These are realized by the Fock space relations
\[ \{b_{q\bar{q}}^+, b_{q\bar{q}}^+\} = \{d_{q\bar{q}}^+, d_{q\bar{q}}^+\} = \delta_{q\bar{q}} \delta^{(3)}(x-x'), \]
\[ [a_q, a_{q'}^*] = \delta^{(3)}_{q, q'}, \]
and
\[ \{b, b\} = \{d, d\} = \{b, d^*\} = \{a, b\} = \{a, d^*\} = 0. \]

It is now straightforward to insert the expansions (5) and (8) into (3), and integrate over an elementary box as shown in (2). The result is given in Appendix B, along with some technical details of the calculation. A few points are worth amplifying, however.

First, the choice of antiperiodic boundary conditions in \( x^- \) for the fermion field results in a considerable simplification within the set of basis functions. Note that the scalar field is chosen to be periodic in \( x^- \) so that, for example, the various terms in the fermion equations of motion all satisfy the same boundary conditions.

Second, there is a choice to be made in defining the antiderivative necessary to solve the constraint equation for \( \psi_- \) (4). We shall define \( \delta_- \) applied to an antiperiodic function to be the unique antiderivative that is itself an antiperiodic function. Specifically,
\[ \psi_-(x) = -\frac{i}{4} \int_{x^-}^{x^- + \pi} dy^- \varepsilon(x^- - y^-) \left[ -i\alpha^+ \partial_+ m\beta + g\beta \phi(y^-, x) \right] \psi_+(y^-, x) \]
with \( \varepsilon(x) \) the antisymmetric step function:
\[ \varepsilon(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \]

From the equation of motion for \( \psi_+ \),
\[ i\partial_+ \psi_+ = \frac{1}{2} \left[ -i\alpha^+ \partial_+ m\beta + g\beta \phi(y^-, x) \right] \psi_-, \]

it follows that the definition (15) of \( \psi_- \) guarantees the antiperiodicity of \( \psi_- \) for all \( x^+ \), assuming of course that \( \phi \) remains periodic throughout its evolution. One thing that (15) does not accomplish, however, is to endow \( \psi_- \) with its expected singular structure in \( x^+ \). That is, the anticommutator of \( \psi_- \) and its hermitean adjoint at equal \( x^- \) should include a delta function, but (15) results in a completely regular anticommutator. In certain cases this singularity is important, and must be introduced by hand if it is not already present [22]. The implications of this missing feature of the light-cone theory are not yet clear. We shall return to these (and related) issues below.

Let us now calculate the lowest-order perturbative correction to the energy of the one-fermion state using the Hamiltonian of Appendix B. There are two contributions: \( \delta P_1 \), coming from one fermion-one boson intermediate states, and \( \delta P_2 \), coming from the self-induced inertia terms in the Hamiltonian. We find
\[ \delta P_1 = -\frac{\alpha L}{2\pi L^2} \sum_{q, q'} \sum_{n=1}^{\infty} \frac{1}{n(n-q)} \left[ n^2 \left( \frac{q-q_n}{n} \right)^2 + (2n-q)^2 \beta_f \right] \]
\[ + \frac{1}{n(n-q)} \frac{n^2}{q-q_n} \left( \frac{q-q_n}{n} \right)^2 + q^2 \beta_f + n(n-q)\beta_b \]
\[ \delta P_2 = -\frac{\alpha L}{4\pi L^2} \sum_{q, q'} \sum_{n=1}^{\infty} \frac{1}{n(n-q)} \left[ \frac{1}{n-q} + \frac{1}{n+q} \right] \]

where \( \left( \frac{\pi n}{L}, \frac{\pi n}{L_\perp} \right) \) is the momentum of the incoming fermion, \( \alpha \equiv \frac{g^2}{4\pi} \), and
\[ \beta_f \equiv \left( \frac{mL_{\perp}}{\pi} \right)^2, \quad \beta_b \equiv \left( \frac{\mu L_{\perp}}{\pi} \right)^2. \]

Each of these contributions diverges like \( A^2 \), where \( A \) is a cutoff in the transverse momentum. In a continuum