SIMILARITY ANALYSIS AND EXACT SOLUTIONS FOR A GENERAL DISCRETE TWO-VELOCITY MODEL OF BOLTZMANN EQUATION*

Carmela Currò**, Francesco Oliveri**

1. INTRODUCTION

The Boltzmann equation is of fundamental importance in kinetic theory of dilute monoatomic gases. It is a non-linear integro-differential equation relating the rate of change of the one particle function in a gas of particles mutually interacting to a collision term describing the binary interactions between the molecules. The very complex mathematical structure of this collision term has been the main obstacle in the research of exact solutions of the Boltzmann equation.

In recent years the discovery, made independently by Bobylev [1] and Krook and Wu [2], of an exact similarity solution of the nonlinear Boltzmann equation for Maxwell molecules caused a great revival of interest in this direction (see [3] and the references therein quoted).

A lot of results have been carried out in the case of spatially homogeneous Boltzmann equation [3], [4] for which theorems of existence and uniqueness as well as solutions have been established. But if the dependence on space variables is taken into account, only local results or global solutions for particular initial data are known [5], [6], [7], [8]; even the existence of solutions is shown under very restrictive conditions [4].

In order to avoid the main difficulties arising by studying the true Boltzmann equation, starting from the pioneeristic work of Carleman [9] many discrete velocity models have been introduced [9], [10], [11].

In these models only a finite number of molecules velocities are possible and the study of their solutions may provide informations about the properties of the original Boltzmann equation.

In this paper we shall consider a discrete two-velocity model first considered by Illner [12] and recently investigated by Platkowski [13] involving a general quadratic collision term:

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (A u) = 0 \]

\[ \frac{\partial v}{\partial t} - \frac{\partial}{\partial x} (A v) = 0 \]

(1.1)

where:

\[ \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \]

\[ A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a, b, c \in \mathbb{R}; \quad (\cdot, \cdot) \text{ is the scalar product in } \mathbb{R}^2 \]

and \( U = (u, v)^T \), \( u \) and \( v \) are the densities of the particles moving in the positive and negative direction of the \( x \)-axis with the velocities \( +1 \) and \( -1 \) respectively.

This model generalizes the well known Carleman [9] and McKean [10] models which are obtained for \( a = -1, b = 0, c = 1 \) and for \( a = 0, b = -1/2, c = 1 \) respectively.

In the next sections our aim will be to apply the similarity methods to the system (1.1) in order to get possible sets of exact solutions. Moreover we will specialize the solutions obtained to the Carleman and McKean models.

Our study is motivated because of the existence in the literature of various attempts in order to construct exact similarity solutions for models of the Boltzmann equation [14], [15].

In particular this analysis has been carried out quite recently by Wick [16] who showed two classes of global explicit solutions of the Carleman model and the conditions to be satisfied in order to show the non negativity of the solutions; moreover Dukek and Nonnemacher [17] in a recent paper presented undiscovered solutions of the Carleman model and classified already known [3], [16] solutions.

2. DETERMINATION OF LIE GROUPS AND ALGEBRA ASSOCIATED

The system (1.1) can be written as follows:

\[ \Sigma = U_t + B \cdot U_x - F = 0 \]

(2.1)

where

\[ B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} (A u, U) \\ -(A u, U) \end{bmatrix} \]

and the subscripts stand for partial derivatives with respect to the variables. Since an exhaustive description of the

---

* Work supported by the C.N.R. through the G.N.F.M.

** Dipartimento di Matematica, Università di Messina, via C. Battisti 90, 98100 Messina (Italy).
determination of the full Lie group and algebra associated may be found in [18], [19], [20], we give here only short features of the procedure to be applied. To determine the most general infinitesimal group leaving invariant the system (2.1) the non-extended generator, involving the original variables \( x, t \) and \( U \) only,

\[
L(\cdot) = \frac{\partial}{\partial x^a} + \nabla \alpha \cdot V
\]  
(2.2)

\( \alpha = 0,1; x^0 = t; x^1 = x; \Gamma^a = \Gamma^a(U, x^2); \)

\[
\nabla = \partial / \partial U; \quad V = V(U, x^0) = (W, Z)^T
\]

is extended, to include the first derivatives of \( U \), to:

\[
L_1(\cdot) = L(\cdot) + \nabla \alpha(\cdot) P^a
\]  
(2.30)

where

\[
\nabla = \partial / \partial U_a;
\]

\[
P^a = \frac{\partial V}{\partial x^a} + \nabla \cdot U_a \cdot \frac{\partial}{\partial x^a} + \nabla \Delta U_a \cdot U_B
\]

In our case the invariance condition writes:

\[
L_1(\Sigma) = 0
\]  
(2.4)

Taking into account (2.1) we have:

\[
- V F \cdot V + V \cdot F + (\nabla V \cdot F - \nabla \Gamma^0 F - (\nabla \Gamma^0 F) F + B \cdot V_x - \nabla \Gamma^0 B \cdot F - (\nabla V \cdot B) U_x + \nabla \Gamma^0 B \cdot U_x + (\nabla \Gamma^0 B \cdot U_x) F - \nabla \Gamma^1 F \cdot U_x + (B \cdot \nabla V) U_x + \nabla \Gamma^0 B \cdot U_x + (B \cdot \nabla V) U_x + (\nabla \Gamma^1 B \cdot U_x) F - (\nabla \Gamma^0 B \cdot U_x) F - B \cdot \nabla \Gamma^1 B \cdot U_x - (\nabla \Gamma^0 B \cdot U_x) F - B \cdot \nabla \Gamma^1 B \cdot U_x + (\nabla \Gamma^0 B \cdot U_x) F - B \cdot \nabla \Gamma^1 B \cdot U_x + (\nabla \Gamma^0 B \cdot U_x) F - B \cdot \nabla \Gamma^1 B \cdot U_x = 0.
\]  
(2.5)

That represents a system of two coupled polynomial of second degree in \( u_x \) and \( o_x \) whose coefficients must vanish since the components of \( U_x \) have to be considered as independent variables. Such a request provides a set of determining equations for the group generators \( \Gamma^a \) and \( V \):

\[
\Gamma^0_v = \Gamma^1_v
\]

\[
\Gamma^0_u = - \Gamma^1_u
\]

\[
W_v = - (A U, U) \Gamma^0_v = 0
\]

\[
2(A U, U) \Gamma^0_v + \Gamma^0_r - \Gamma^1_r + (\nabla \Gamma^0 U_x) = 0
\]  
(2.6)

\[
Z_v + (A U, U) \Gamma^0_u = 0
\]

\[
2(A U, U) \Gamma^0_v - \Gamma^0_r - \Gamma^1_r + (\nabla \Gamma^0 U_x) = 0
\]

\[
(A U, U) \cdot [W_v - \Gamma^0_r - \Gamma^0_x - (A U, U) \Gamma^0_u] + W_x + W_v - 2(A U, U) = 0
\]

\[
(A U, U) \cdot \Gamma^0_v - \Gamma^0_r - Z_v - (A U, U) \Gamma^0_u + Z_r - Z_x + 2(A U, U) = 0
\]

Essentially three cases must be considered for integrating the system (2.6), depending on the choice of the constants \( a, b, c \) of the matrix \( A \).

\[\text{Case 1: } a, b, c \text{ such that } b^2 - ac \neq 0 \text{ and } (a \neq 0 \text{ or } c \neq 0).\]

In this case the integration of the system (2.6) leads to the following expression for the generators of Lie group:

\[
\Gamma^0_r(t) = \alpha t + \beta
\]

\[
\Gamma^1_x = \alpha x + \gamma
\]  
(2.7)

\[
W(u) = - \alpha u
\]

\[
Z(v) = - \alpha v.
\]

Therefore three main infinitesimal transformation groups are characterized corresponding to the arbitrary constants \( \alpha, \beta, \gamma \). The condition \( b^2 - ac \neq 0 \) in particular is verified for the Carleman and for the McKean model, consequently, the infinitesimal group found is valid in both models.

\[\text{Case 2: } a, b, c \neq 0 \text{ such that } b^2 - ac = 0.\]

This choice of the constants \( a, b, c \) allows us to get for the Lie group:

\[
\Gamma^0_r(t) = \alpha t + \beta
\]

\[
\Gamma^1_x = \alpha x + \gamma
\]  
(2.8)

\[
W(u) = - \alpha u + \delta
\]

\[
Z(v) = - \alpha v - (a/b) \delta
\]

where \( \alpha, \beta, \gamma, \delta \) are arbitrary constants.

The relations (2.7) characterize, depending on the choice of the arbitrary constants, time translation, space translation or the stretching group; the relation (2.8) in addition to the above mentioned groups of invariance characterize also translation of the dependent variables \( u \) and \( v \).

\[\text{Case 3: } a = c = 0, b \neq 0.\]

In this case the generators of the infinitesimal transformation group become:

\[
\Gamma^0_r(x, t) = f(x + t) - g(x - t)
\]

\[
\Gamma^1_x = f(x + t) + g(x - t)
\]  
(2.9)

\[
W(u, t, u) = - 2 g^\prime(x - t) \cdot u
\]

\[
Z(x, t, u) = - 2 f^\prime(x + t) \cdot u
\]

where \( f \) and \( g \) are arbitrary functions of \( x + t \) and \( x - t \), respectively and the primes denote differentiation with respect to the arguments.

We remark here that (2.9) originate a Lie group having an infinite dimensional algebra as the generators depend on two arbitrary functions.

3. CALCULATION OF INVARIANT SURFACES

Once the infinitesimal group leaving invariant the system (2.1) has been determined we have to consider the invariant surfaces which are solutions of the following system of partial differential equations:

\[
\Gamma^0 U_x + \Gamma^1 U_x = V
\]  
(3.1)

The integration of (3.1) is made by solving the correspond-