LINEAR ELASTIC FRACTURE MECHANICS AS A LIMIT - CASE OF STRAIN - SOFTENING INSTABILITY*

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SUMMARY. Size-scale, slenderness and degree of redundancy are shown to have a fundamental impact on the global structural behaviour, which can range from ductile to brittle when strain softening and localization are taken into account. The brittle behaviour involves an internal instability in the load-deflection path, which shows a positive slope in the softening branch. This virtual branch is revealed only if the loading process is controlled by a monotonically increasing time function (e.g., the crack opening displacement). Otherwise, the loading capacity shows discontinuity with a negative jump. When the post-peak behaviour is kept under control up to the complete structure separation, the area bounded by the load-deflection curve and the deflection axis represents the energy dissipated in the fracture process. A general explanation of the well-known decrease in apparent strength and increase in fracture toughness by increasing the specimen size is given in terms of dimensional analysis. Due to the different physical dimensions of strength [FL^{-2}] and toughness [FL^{-1}], the actual values of these two intrinsic material properties may be found exactly only with comparatively large specimens.

1. INTRODUCTION

The curvature localization for bent beams is related to the fracture toughness of the material. Even if no initial cracks are assumed to exist in the structure, the fracture mechanics concepts are applicable. Size-scale, slenderness and degree of redundancy are shown to exert a fundamental influence on the global structure behaviour, which can range from ductile to brittle when softening is taken into account.

When a bifurcation (or internal instability) of the load-deflection equilibrium path occurs, the softening slope becomes positive. If the loading process is deflection-controlled, the loading capacity of the beam shows discontinuity with a negative jump. This dreadful event is favoured by large size-scale, high slenderness and/or low degree of redundancy [1].

Strain localization and softening are then considered in order to analyze the three-point bending of rectangular slabs. Even in this case, for large sizes and high slendernesses, the slope of the global softening branch appears to be positive. This branch is not virtual only if the loading process is controlled by a monotonically increasing time function. When the post-peak behaviour is kept under control up to the complete structure separation, the area outlined by the load-deflection curve and the deflection axis represents the product of fracture toughness ($G_{IC}$) by the initial cross-section area.

An explanation of the well-known decrease in apparent strength by increasing specimen size is given, without making the usual statistical Weibull assumptions and assuming the existence of initial cracks and defects. It has been proved that the actual strength of the material can be obtained only with specimens of infinite size, where the influence of heterogeneity becomes unimportant. With the real specimens, an apparent strength higher than the actual one is consistently found.

A cohesive crack model is eventually applied in order to analyze the three-point bending of initially cracked rectangular slabs. Once more, an internally unstable post-peak behaviour occurs for large sizes. This may be kept

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under control by the crack mouth opening displacement, which is an increasing function of time, even when load and deflection decrease following the softening branch with positive slope. The area marked out by the load-deflection curve and the deflection axis is the product of fracture toughness ($G_{IC}$) by initial ligament area.

An explanation of the recurrent experimental increase in fracture toughness by increasing specimen size is provided, without resorting to the usual non-linear theories and the energy dissipation in the crack tip plastic zones. It has been proved that the actual fracture toughness ($K_{IC}$) of the material can be obtained only with specimens of infinite size. With the real specimens, a fictitious fracture toughness lower than the actual one is consistently measured [2].

2. STRAIN-SOFTENING INSTABILITY OF BENT BEAMS

Let us consider a linear elastic moment-curvature relation (Fig. 1-a) associated with a linear softening moment-rotation law (Fig. 1-b). The area below the moment-rotation curve represents the energy dissipated by the softening hinge until this becomes a free-rotation hinge without resisting moment. This area is equal to the product of the separation energy or strain energy release rate, $G_{IC}$, by the cross-section area, $A$:

$$\frac{1}{2} M_u \phi_0 = G_{IC} A. \quad (1)$$

For the statically determined beam in Fig. 2-a the central deflection is:

$$\delta = \frac{23}{648} \frac{P h^3}{E I}, \quad (2)$$

$E$ being the Young's modulus of the material and $I$ the inertial moment of the cross-section of the beam. If the cross-section is assumed to be rectangular with depth $b$ and thickness $t$, the ultimate strength load $P_u$ is given by:

$$M_u = \frac{P_u t b^2}{6} = P_u \frac{K}{3}, \quad (3)$$

where $\sigma_u$ is the ultimate tensile strength of the material. From eqs. (2) and (3) it is possible to obtain the deflection in the ultimate strength condition:

$$\delta_u = \frac{23}{108} \frac{\sigma_u b^2}{P_u}, \quad (4)$$

where $\sigma_u = \sigma_u / E$.

If, in the ultimate strength condition, two softening hinges form at the load application points, the bending moments in the hinges and, generally, in the beam decrease. At this intermediate stage, the hinge rotations increase, whereas the beam curvatures decrease. Central deflection increases or decreases with the prevalence of hinge rotations or beam curvatures respectively. Eventually, when hinge rotations achieve the limit value $\phi_0$, the beam is completely unloaded with $P = 0$ and $\delta = \delta_0 = \phi_0 / 3$ (Fig. 2-b).

Rearranging of eq. (2) gives:

$$P = \frac{648}{23} \frac{E I}{k^3} \delta, \quad \text{for} \; \delta < \delta_0, \quad (5)$$

while the condition of complete load relaxation reads:

$$P = 0, \quad \text{for} \; \delta \geq \delta_0. \quad (6)$$

When $\delta_0 > \delta_u$, the softening process is stable only if deflection-controlled, since the slope $dP/d\delta$ is negative (Fig. 3-a). When $\delta_0 = \delta_u$, the slope $dP/d\delta$ is not defined and a drop in the loading capacity occurs, even if the loading is deflection-controlled (Fig. 3-b). Eventually, when $\delta_0 < \delta_u$, the slope $dP/d\delta$ becomes positive (Fig. 3-c) and the same negative jump occurs as that shown in Fig. 3-b. Therefore internal instability occurs for [3]:

$$\frac{\phi_0 l}{3} \leq \frac{23}{108} \frac{\sigma_u b^2}{P_u}, \quad (7)$$

i.e., for:

$$\frac{\phi_0 l}{3} \leq \frac{23}{\sigma_u \lambda}, \quad (8)$$

where $\lambda = l/b$ is the beam slenderness. Recalling eqs. (1) and (3), the following is obtained:

$$\phi_0 = 12 \frac{G_{IC}}{\sigma_u b} = 12 s_{E}, \quad (9)$$

where:

$$s_{E} = \frac{G_{IC}}{\sigma_u b} \quad (10)$$