One solves the following problem of M. V. Keldysh: let $\mathcal{H}$ be a completely continuous self-adjoint operator acting in a separable Hilbert space $\mathcal{K}$, $K_0\mathcal{H} = \{0\}$, $T = \mathcal{H}(I+S)$ being a weak perturbation (i.e., the operator $S$ is completely continuous and $I+S$ is invertible); is it true that the operator $T$ will be complete together with $\mathcal{H}$ (i.e., the family of its root vectors complete in $\mathcal{H}$)? The answer is negative. One describes all operators, for which the answer is positive (for any $S$): these are those totally positive completely continuous operators $\mathcal{H}$ for which

$$\lim_{t \to \infty} \frac{\log[V(t)(1+\varepsilon)] - V(t+1)}{\log t} < +\infty,$$

where $V(t)$ is the number of eigenvalues of $\mathcal{H}$ larger than $\frac{1}{t}(t>0), \varepsilon>0$.

We shall say that a linear operator, acting in a separable Hilbert space $\mathcal{K}$, is total, if the system of its root vectors, corresponding to the nonzero eigenvalues, is total in $\mathcal{K}$. By a weak perturbation of a total completely continuous self-adjoint operator $\mathcal{H}$ we shall mean the operator $\mathcal{H}(I+S)$, where $S$ is such a completely continuous operator that the operator $I+S$ is continuously invertible.

Keldysh [1] has established that any weak perturbation of a total self-adjoint operator $\mathcal{H}$, for which the series $\sum |\lambda_j|^p$, $p>0$, converges, where $\{\lambda_j\}$ is the collection of all eigenvalues (taking into account their multiplicities), is a total operator. He has formulated the following problem. Is there a total completely continuous self-adjoint operator whose weak perturbation is not total? We shall answer this question in the affirmative, indicating (Theorem 1) a broad class of operators for which the weak perturbation is not only nontotal but it is a Volterra operator.* In addition, we give an exhaustive description (in terms of the spectra) of all total completely continuous positive operators all of whose weak perturbations are total.

*A completely continuous operator is said to be a Volterra operator if its spectrum is concentrated at the point 0.

THEOREM 1. Assume that the eigenvalues \( \{\lambda_j\}_{j=1}^{\infty} \), enumerated in decreasing order and taking into account their multiplicities, of a total positive completely continuous operator \( H \) satisfy the condition

\[
\lim_{n \to \infty} \lambda_{2n} \cdot \lambda_n^{-1} = 1.
\]

Then there exists a weak perturbation of the operator \( H \) which is a Volterra operator.

THEOREM 2. Let \( H \) be a total positive completely continuous operator and let \( \nu(t) \) be the number of their eigenvalues (taking into account their multiplicities) larger than \( 1/t \). In order that all the weak perturbations of \( H \) be total operators it is necessary and sufficient that there exist a number \( \varepsilon > 0 \), such that

\[
\lim_{t \to \infty} \frac{\log [\nu(t(1+\varepsilon))-\nu(t)+1]}{\log t} < +\infty.
\]

THEOREM 3. Assume that a normal completely continuous operator \( H \) can be represented in the form

\[
H = \lambda_1 H_1 \oplus \ldots \oplus \lambda_m H_m,
\]

\( |\lambda_i| = 1 \), \( H_i (1 \leq i \leq m) \) being a total positive operator acting in the space \( \mathcal{H}_i \), \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_m \). In order that every weak perturbation of the operator \( H \) be a total operator it is necessary and sufficient that condition (1) be satisfied for each operator \( H_i \).

Our paper consists of three sections. In the first two sections we prove Theorems 1 and 2 and in Sec. 3 we carry out the proofs of the auxiliary statements. We do not give the proof of Theorem 3 since it is completely similar to the proof of Theorem 2. The results of the present paper appeared in the dissertation of one of the authors [2]. An outline of the proof of Theorem 1 is contained in [3].

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1. Proof of Theorem 1

The following lemma allows us to consider from the very beginning operators \( H \) with a regular behavior of the eigenvalues.

LEMMA 1.1. An operator \( H \), satisfying the conditions of Theorem 1, admits a weak perturbation which is a total positive operator, whose eigenvalues \( \{a_n\}_{n=1}^{\infty} \) possess the following properties

\[
a_n = a_n > 0; \ a_n \downarrow 0 (n > 0); n(a_n - a_{n+1}) = o(a_n), n \to \infty.
\]

We consider now in the space \( \mathcal{H} \), which we consider realized as \( L^2(0, 2\pi) \), a total positive completely continuous operator \( \mathcal{P} \), whose sequence of eigenvectors is the system \( \{e^{inx}\}_{n=-\infty}^{\infty} \); while the corresponding sequence of eigenvalues is the sequence \( \{a_n\}_{n=1}^{\infty} \), possessing the properties (1.1). This operator is unitarily equivalent to the operator constructed in Lemma 1.1. We define now a completely continuous self-adjoint operator \( Q \) in the following manner: a total orthonormal sequence of eigenvectors of the operator \( Q \) is the system \( \{e^{i(n+\alpha)x}\}_{n=-\infty}^{\infty} \), and the corresponding eigenvalues \( \{b_n\}_{n=1}^{\infty} \) are related with the eigenvalues of the operator \( \mathcal{P} \) by the Riesz–Titchmarsh transform.