Small Perturbations of $C^*$-Dynamical Systems

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Abstract. It is shown that if $\delta$ is the generator of a strongly continuous one-parameter group of *-automorphisms of a $C^*$-algebra $A$ and $\delta'$ is an unbounded *-derivation of $A$ with the same domain as $\delta$, then $\delta + \varepsilon \delta'$ is also a generator for all sufficiently small real numbers $\varepsilon$.

The perturbation theory of strongly continuous one-parameter contraction semigroups $\{e^{\varepsilon t} : t \geq 0\}$ on Banach spaces shows that several features of these systems are stable under relatively bounded perturbations [6, 8]. For example if $T'$ is a dissipative operator with the same domain $\mathcal{D}$ as $T$, then $T + T'$ is the generator of some contraction semi-group, provided that

$$\|T'x\| \leq \alpha \|x\| + \beta \|Tx\|$$

for all $x$ in $\mathcal{D}$, for some constants $\alpha$ and $\beta < 1$.

In the $C^*$-algebraic model of a quantum dynamical system, the time evolution is represented by a strongly continuous one-parameter group of *-automorphisms $\{e^{\iota t} : t \in \mathbb{R}\}$ of a $C^*$-algebra $A$, where the generator $\delta$ is a closed unbounded *-derivation. Longo [7] has shown that in this case, any *-derivation $\delta'$ with the same domain is automatically relatively bounded with respect to $\delta$. In this note it will be shown that $\delta'$ is also necessarily dissipative, and therefore $\delta + \varepsilon \delta'$ is a generator for all sufficiently small $\varepsilon$ (cf. [4, Sect. 5]).

Longo's result also applies if $\delta$ is any closed *-derivation (not necessarily a generator), and he asked whether $\delta'$ is then necessarily closable. For commutative $C^*$-algebras, an affirmative answer to this problem was given in [3, Theorem 5.3]. The proof there involved showing that any (maximal) closed ideal containing $a$ and $\delta(a)$ also contains $\delta'(a)$. Since the maximal ideals in a commutative $C^*$-algebra are of codimension 1 and have zero intersection, this enabled a very specific description of $\delta'$ to be given in terms of $\delta$. For non-commutative $C^*$-algebras, it will be shown here that $\delta'(a)$ again belongs to the closed ideal generated by $a$ and $\delta(a)$, and a partial answer to Longo's question will be given. All the results of this
note are obtained as corollaries of Theorem 3, the proof of which makes use of Longo's theorem, functional calculus in the domain of $\delta$ and the Hahn-Banach separation theorem.

Let $\delta$ be an (unbounded) $\ast$-derivation of a $C^*$-algebra $A$, defined on a dense $\ast$-subalgebra $\mathcal{D}$ of $A$. We recall the following definitions from [1, 10, 11]. An operator $a$ in the self-adjoint part $\mathcal{D}^\ast$ of $\mathcal{D}$ is $\delta$-well-behaved (resp. strongly $\delta$-well-behaved) if $\phi(\delta(a))=0$ for some (resp. for all) states $\phi$ of $A$ with $|\phi(a)|=\|a\|$. The derivation $\delta$ is well-behaved if every operator in $\mathcal{D}^\ast$ is $\delta$-well-behaved; $\delta$ is quasi well-behaved if there is a dense open subset of $\mathcal{D}^\ast$ (in the relative topology) consisting of $\delta$-well-behaved operators. A closed ideal $J$ in $A$ is $\delta$-invariant if $\delta$ maps $\mathcal{D}\cap J$ into $J$. Then $\delta$ induces a $\ast$-derivation $\delta_J$ of $A/J$ with dense domain $\pi_J(\mathcal{D})$, given by

$$\delta_J(\pi_J(a))=\pi_J(\delta(a)) \quad (a \in \mathcal{D})$$

where $\pi_J$ is the quotient mapping of $A$ onto $A/J$. The derivation $\delta$ is pseudo well-behaved if for each non-zero $a$ in $A$ there is a $\delta$-invariant ideal $J$, not containing $a$, such that $\delta_J$ is quasi well-behaved. Note that the generator of any one-parameter $\ast$-automorphism group is well-behaved.

The following lemma is known, but since different authors have used different, but equivalent, definitions and terminology, we state it here for convenience.

**Lemma 1.** Let $\delta$ be a $\ast$-derivation of $A$ with dense domain $\mathcal{D}$. The following are equivalent:

(i) $\delta$ is well-behaved.

(ii) For any $a$ in $\mathcal{D}$, there is a non-zero functional $\phi$ in $A^\ast$ such that $\phi(a)=\|a\| \|\phi\|$ and $Re\phi(\delta(a))=0$.

(iii) $Re\phi(\delta(a))=0$ for any $a$ in $\mathcal{D}$ and $\phi$ in $A^\ast$ such that $\phi(a)=\|a\| \|\phi\|$.

(iv) Every operator in $\mathcal{D}^\ast$ is strongly $\delta$-well-behaved.

(v) $\|a+a\delta(a)\| \geq \|a\|$ for all $a$ in $\mathcal{D}$ and $a$ in $\mathcal{D}$.

**Proof.** The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) were proved in [11, Proposition 2.19] and [2, Corollary 3] respectively, (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are trivial, and (ii)$\Rightarrow$(v) follows from [5, Theorem V.9.5].

**Lemma 2.** Let $J$ be a closed ideal in $A$ and $\mathcal{D}$ be the dense domain of a closed $\ast$-derivation of $A$. Then $J\cap \mathcal{D}$ is dense in $J$.

**Proof.** Let $a$ be a self-adjoint operator in $J$. For any $\epsilon>0$, there exists $b$ in $\mathcal{D}^\ast$ such that $\|a-b\|<\epsilon/3$. Let $f$ be a $C^2$-function on $\mathbb{R}$ such that

$$|f(t)-t|<\frac{2\epsilon}{3} \quad (t \in \mathbb{R})$$

$$f(t)=0 \quad (|t| \leq \frac{\epsilon}{3}).$$