The Scattering of Sound Waves by a Cone.

By

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The problem of the disturbance produced by a Point-Source of Sound in an Infinite Medium containing a Rigid Obstacle has been solved for the case when the obstacle is a Sphere*, and when it consists of two planes intersecting at any angle**. This paper contains the solution of the case when the obstacle is a Right Circular Cone of any angle.

§ 1.

The Spherical Polar Coordinates $(r, \theta, \varphi)$ are used, and the cone is given by $\theta = \theta_0$. The region occupied by the medium is defined by

$$0 < r < \infty,$$

$$0 < \theta < \theta_0,$$

$$0 < \varphi < 2\pi.$$

We require a solution of the equation

$$(1) \quad \nabla^2 u + k^2 u = 0,$$

which, together with its first and second derivatives, is finite and single-valued throughout this region, except at the point $(r', \theta', \varphi')$, where the source is situated.


**) If the angle is $\frac{\pi}{m}$, $m$ any positive integer, the ordinary method of images gives the solution. If the angle is $\frac{n\pi}{m}$, $m$ and $n$ any positive integers, the method of images in a Riemann's Space is applied. [Carslaw, Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications, Proc. Lond. Math. Soc. (1) 30, p. 155 (1899)]. In a later communication [Proc. Lond. Math. Soc. (2) 8, p. 365 (1912)] I have pointed out that a suitable solution of the equation of period $2\alpha$ leads at once to the solution of the problem for the case of two planes intersecting at any angle $\alpha$. 
At \((r', \theta', \varphi')\), \(u\) is to become infinite as

\[
e^{-ikR} \frac{e^{-ikR}}{R},
\]

when \(R\) tends to zero.

At the surface of the cone \(\theta = \theta_0\), we have the condition

\[
\frac{\partial u}{\partial n} = 0 \quad \text{(i. e. } \frac{\partial u}{\partial \theta} = 0)\]

And since there is no reflection at infinity\(^\ast\), \(u\) cannot involve terms of the type

\[
e^{ikr},
\]

when \(r\) tends to infinity.

\section*{§ 2.}

I shall examine first the case when the source is situated on the axis of the cone, so that \(\theta' = 0\).

From symmetry it follows that \(u\) does not involve \(\varphi\), and equation (1) takes the form

\[
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial u}{\partial \mu}\right) + k^2 u = 0 \quad (\mu = \cos \theta).
\]

If the cone were absent and the medium filled all space, the solution would be

\[
u_0 = \frac{e^{-ikR}}{R},
\]

where

\[
R = \sqrt{r^2 + r'^2 - 2rr'\mu}.
\]

It is known that, with this notation,\(^\ast\ast\)

\(^\ast\) Pockels called attention [in his book \(\text{"Uber die partielle Differentialgleichung } \Delta u + k^2 u = 0 \text{ und deren Auftreten in der Mathematischen Physik, p. 305}\)] to the fact that the analytical problem is indeterminate in the case of an infinite region unless some condition is added of this nature.

Compare also a paper by Sommerfeld in Jahresb. D. Math. Ver. 22 (1913), entitled \(\text{Die Greensche Funktion der Schwingungsgleichung.}\)

\(^\ast\ast\) Cf. Heine, \(\text{Handbuch der Kugelfunktionen, Bd. 1, p. 346; Macdonald, Proc. Lond. Math. Soc. (1) 32, p. 157 (1900). In this paper }\)

\[
K_n(ix) = \frac{\pi}{2 \sin n\pi} e^{-\frac{1}{2} \pi i n} (J_n(x) - e^{n\pi i} J_n(x))
\]

is taken as the Second Solution of Bessel's Equation of the \(n\)th order.