Abstract. Based on a local Gaussian evaluation of the functional integral representation, a method is developed to obtain ground state functionals. The method is applied to the gluon sector of QCD. For the leading term in the ground state functional, stochastic techniques are used to check consistency of the quantum theory, finiteness of the mass gap and the scaling relation in the continuum limit. The functional also implies strong chromomagnetic fluctuations which constrain the propagators in the fermion sector.

1 Introduction

There is a wide belief that quantum chromodynamics is the theory of strong interactions. However reliable predictions can be extracted from the theory only for a restricted class of experimentally observable quantities. For short-distance (large transverse momentum) phenomena, perturbation theory seems reliable and, for the static properties of hadrons, a few results have been obtained by lattice Monte Carlo calculations. However high energy small transverse momentum reactions, for example, are out of reach of both techniques.

Due to asymptotic freedom the (perturbative) physical picture of large transverse momentum reactions is well understood. For low energies however, even if future dedicated machines get around the small lattice limitation and are eventually able to crunch out the right numbers, some sort of understanding of the physical picture at low energies would be convenient. The physical picture is probably in the tapes containing thousands of computer-generated lattice configurations, but this is not very transparent for the common mortal.

The construction of approximations to the vacuum and other low-lying states in QCD has been attempted by many authors [1–10] who used several approaches and conjectures. An approach that has been widely followed is the variational one, wherein a trial state is conjectured, dependent on some parameters, which are then adjusted to minimize the energy. However, as Feynman [11] has pointed out, although the trial states are constructed to describe the low energy properties, the non-linear nature of the QCD couplings (and the large number of degrees of freedom) have as a consequence that the variational parameters tend to adjust themselves to the high frequency modes because they have the largest contribution to the energy. This in general leads to unrealistic values of the parameters as far as the low energy properties are concerned, which are exactly what we were trying to describe to begin with. In addition because one needs to compute high dimensional energy integrals, this essentially restricts the trial states to be Gaussian. These two limitations seriously restrict the usefulness of the variational approach in field theory.*

In other approaches a systematic perturbative expansion of the vacuum is constructed where typically the first term has the structure of the Abelian ground state. In the perturbative expansions the individual terms are not invariant under non-Abelian gauge transformations and non-perturbative features are hard to extract from the expansion. It has been proposed [6] to improve the perturbative expansion by a so-called gauge invariant completion. i.e. order by order the expansion terms are replaced by gauge-invariant expressions which agree, to that order, with perturbation theory. A difficult consistency problem occurs however, because the gauge-invariant completion of a given order contains contributions of all higher orders.

The general conclusion is that methods to approximate the QCD vacuum are needed which do not rely on the minimization of the energy and that are gauge invariant. Furthermore they should also not rely on expansions on powers of the coupling constant to ensure that the leading terms already contain non-perturbative information. An attempt in this direction is presented in Sect. 2.

* A different point of view has recently been advocated by A. Neveu [12]
based on a local Gaussian evaluation of the functional integral representation of the ground state.

When an approximation to the QCD vacuum is derived or conjectured, the second problem is what to do with it. In principle when the ground state is known, Green's functions and, by LSZ reduction, scattering amplitudes may be computed and the theory is completely defined. However one still has to compute functional integrals on the time-zero fields which, for non-trivial (non-Gaussian) ground states is not simple.

There is however another way to extract useful information from the ground state functionals \( \psi_0(\phi) \) that relies on the interpretation of the measure \( \psi_0(\phi) \{ d\phi \} \) as the invariant measure of a stochastic process

\[
d\phi = -L(\phi)\,dt + dW(t),
\]

with drift

\[
-L(\phi) = \frac{1}{\psi_0(\phi)} \frac{\delta \psi_0}{\delta \phi},
\]

the generator of this process being the Hamiltonian operator. Consistency of the quantum theory defined by the ground state measure may then be rigorously formulated as a problem of closability of the associated energy form [10] or as a problem of existence of solutions of the stochastic differential equation (1.1) [13].

Furthermore the Dirichlet problem of the generator (i.e. the eigenvalue problem of the Hamiltonian) is related to the statistics of the exit time of the stochastic process from the domain where boundary conditions are imposed. In particular the Wentzell-Freidlin theory [14] and its associated large deviation estimates are especially appropriate to characterize the mass gap and scaling properties of the continuum limit. The Wentzell-Freidlin technique seems in fact to be the most natural non-perturbative technique in the sense that quantities behaving like \( \exp(-c/L^2) \) are the simplest ones to deal with. For details on this technique I refer to [14-17]. A summary of the main results and some applications may be found in [18].

Extensive use will be made of the ground state functional interpretation as the invariant measure of a stochastic process in the study of the properties of the ground state approximation constructed in Sect. 2.

The plan of the paper is the following. In Sect. 2 a method is developed to approximate the ground state functional which is then applied to the gluon sector of QCD. One of these approximations, which will be called QCD0, is then shown to lead to a well-defined quantum theory, in the sense that the ground state functional is the density of a closable Dirichlet form [19].

In Sect. 3 the interpretation of the ground state functional as the invariant measure of a stochastic process and the Wentzell-Freidlin technique are used to prove that the long-range non-abelian QCD0-dynamics has a finite mass gap and to derive its scaling properties in the continuum limit. Finally Sect. 4 deals with the structure of the strong chromomagnetic vacuum fluctuations implied by the functional of QCD0 and its effect on the propagation of fermions. Many results in the paper hold for any compact gauge group. However, whenever specific calculations were required, I have, for simplicity, used the \( SU(2) \) group.