The generation of the $\rho$-resonance by QCD

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Abstract. By showing that the imaginary part of a suitable QCD amplitude, after extrapolation up to the cut, exhibits indeed a prominent bump structure where the $\rho$-resonance is expected to be, a rather direct indication for the generation of the $\rho$-resonance by QCD is given. This is achieved by using a mathematically rigorous method of stable analytic extrapolation, based on the theory of maximally converging sequences of polynomials and the application of conformal mappings.

1 Introduction

After the pioneering work of Shifman, Vainshtein and Zakharov (SVZ) [1] it is by now a well known fact that the inclusion of nonperturbative terms to perturbative QCD amplitudes allows at least the determination of the parameters of a number of hadronic resonances. In this context the connection between asymptotic QCD amplitudes and hadrons is made by methods usually called QCD sum rules [1-5].

One of the most striking and impressive results was the determination of the $\rho$-resonance parameters by the above-mentioned authors [1]. In order to appreciate these results properly, one has to realize that the determination of low-energy parameters from asymptotic QCD amplitudes, i.e. amplitudes which typically are strictly valid only in the limit of infinitely large spacelike momenta, is not as straightforward as one may think. This is due to the fact that obviously all different types of QCD sum rules have to use a kind of analytic continuation of error-affected (theoretical QCD) amplitudes, which is a standard example of an ill-posed problem in the sense of Hadamard. This means practically that even very small variations of the theoretical amplitude in the domain of its validity (the domain of large spacelike momenta), which in our case due to the uncertainties in the QCD amplitudes are always allowed, may lead to almost arbitrary variations of the analytic continuation outside the original data domain [6]. This has as a consequence a just as arbitrary variation of the determined hadronic parameters. Being aware of this fact, a QCD sum rule can be a meaningful procedure only if it includes also an appropriate stabilizing prescription. At least for the case of two-point functions there exist such sum rules, which are rigorously proven to be stable [7] (see also [3]), so that the question of stability in the determination of hadronic resonance parameters may be considered as more or less solved.

But there still remains another question, which in a sense is more fundamental than the determination of resonance parameters and which should be investigated before one tries to establish such parameters:

"Is unbiased QCD at all able to generate the existing hadronic resonances?"

This question has not really been answered till now, although the parameters of a series of resonances have been determined in rather good agreement with experiment. The reason for this is the following. The choice of a stabilizing prescription always corresponds to the introduction of some appropriate supplementary information besides the given QCD amplitudes and the information about their domain of analyticity. And this choice of stabilizing supplementary information is indeed a crucial point, since a bad supplementary information, even if it guarantees the stability of the method, may lead to a completely wrong result. When determining resonance parameters, "stability" is typically achieved by forcing the QCD calculations within the full scope (which is given by their errors) to fit as well as possible on a seemingly plausible resonance parametrization.* Therefore the

* "Stability" hereby at first only means the naive and in practice useless stability in the scope of a given fixed parametrization (in finite dimensional (parameter) spaces instabilities cannot occur), and not yet stability with respect to variations of the parametrization (cf. [8]).
stabilizing assumption is that there actually exist resonances within QCD. But then one must not overlook that by doing such fits one always finds an optimal set of parameters for a chosen parametrization, also if in reality there is not even a trace of a resonance in the data! Hence one should be careful with the interpretation of results which are stabilized by using parametrizations. All what can be said with a good conscience is, that if there are really resonances hidden in the theoretical data, then the set of optimal parameters found by the fit should describe these not too badly. Therewith the application of such “fit-methods” is of course very reasonable, but should be complemented by the application of other methods which are able to make sure the existence of resonances without introducing them by hand quite from the beginning.*

For the case of QCD the fact that the obtained resonance parameters agree rather well with experiment should not be overestimated either. Since these results depend heavily on the values of the vacuum condensates, which are by no means uncontroversial (see for example [9]), they should be looked upon as only a first and rather indirect hint for the existence of resonances within QCD.

The aim of this paper is now to give a rough description of a method which, even if it does not completely satisfy our wishes, i.e. it will not render a final proof for the existence of resonances, it is able to yield a rather direct indication for their existence. The method was developed independently by Cutkosky and Deo [10], and Ciulli [11]. It is based on the theory of maximally converging sequences of polynomials (MCSOPS) developed by Walsh and Sewell [12] and the ideas of conformal mappings, which were used in the context of particle physics the first time in the early sixties by many different authors (see for example [13]).

In Sect. 2 we sketch the method, in Sect. 3 we present and discuss the results of its application to the case of the $\varphi$-meson, and in Sect. 4 we give some conclusions.

2 Analytic extrapolation with the help of maximally converging sequences of polynomials

In our application we will be confronted with the following typical situation (Fig. 1): A physical amplitude $\Pi$ is known to be real analytic in the complex cut plane and is explicitly given in a validity domain $H$ (in our special case this will be the domain of validity of the QCD amplitude or for practical reasons a part of it, see below). Now, starting from this validity domain, we would like to perform the analytic continuation of $\Pi$, specifically using the simplest analytic functions, polynomials. It is surely not suitable to try a Taylor-expansion since our information is given in an extended domain (not just in a single point) and, due to the errors, mainly concerns the amplitude $\Pi$ but not its derivatives. This means that essentially we are looking for a kind of generalization of the well known Taylor expansion, in which a single point $s_0$, as the starting domain of a Taylor series*, is replaced by the extended domain $H$.

In order to work on a solid base, one has first to consider the question of the convergence behaviour of such generalized sequences. It is clear that one cannot expect the simple convergence behaviour of a Taylor series, but one has rather to take into account that the convergence behaviour depends strongly on the shape of the domain $H$, and of course also on the prescription how the expansion coefficients -- for a Taylor series these are the derivatives of the amplitude $\Pi$ at $s_0$ -- have to be determined. If for example we want to determine the expansion coefficients $c_n$ of the polynomial

$$S^n(s) = \sum_{k=0}^{n} c_n^k s^k$$  \hspace{1cm} (1)

by minimizing the line integral

$$\oint_{\partial H} |s| |\Pi(s) - S^n(s)|^2$$  \hspace{1cm} (2)

(this is a very simple and convenient prescription), the so-obtained polynomials $S^n$, $n \in \mathbb{N}_0$, are called Szegő polynomials of $\Pi$ in $H$, for which the Walsh-Sewell theory [12] tells us that

1. the set of convergence domains of sequences of Szegő polynomials with respect to all (at least in $H$) analytic functions is fixed alone by the shape of $H$. Each of these domains is bounded by an “equipotential line” $\phi_R$ (cf. Fig. 1) which is defined as the image of the circle with radius $R$ ($R > 1$) under a special conformal mapping $\Phi$ of the exterior of the unit disk onto the exterior of $H$. There exists a nice intuitive interpretation of the curves $\phi_R$.

* This in fact has already been stated by SVZ (cf. Sect. 3.6 in the second of the references [11])

* In the sense that the series is completely determined by the values of the amplitude and its derivatives in this point