Bilocal effective theory with the instantaneous funnel interaction and its renormalization

R. Horvat, D. Kekez, D. Palle, D. Klabučar

1Rudjer Bošković Institute, P.O.B. 1016, 41001 Zagreb, Croatia
2Physics Department of Zagreb University, P.O.B. 162, 41001 Zagreb, Croatia

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Abstract. The bound-state problem for the pion as a quarkonium with the funnel (Coulomb-plus-linear) interaction is solved in a framework that combines the bilocal approach to mesons with the covariant generalization of the instantaneous-potential model. The potential interaction leads to dynamical breaking of chiral symmetry. However, the Coulomb potential leads to ultraviolet divergences that must be subtracted. A careful choice of the renormalization prescription is needed in order to get the correct chiral limit. The mass, the lepton decay constant of the pion, as well as the pion decay width in two photons are calculated.

I Introduction

The activity in the field of relativistic bound states has recently increased. References [1–6] are just some of examples. These papers were motivated, among other things, by the desire to formulate a covariant treatment for quarks interacting through a potential which would hopefully mimic nonperturbative QCD.

Our work is a continuation of the line of research where interquark interactions were modeled by the instantaneous potential. For the case of light quarks, [7–9] may serve as paradigmatic examples of such calculations. A related approach, using the Nambu-Jona-Lasinio model, can be exemplified by [10].

The successes of the potential model in describing heavy quarkonia are well known. For light quarks, however, besides many important qualitative successes, the potential model also exhibited weaknesses: first, the uncertainty concerning the question what potential can provide a realistic and yet reasonably tractable interaction without many free parameters; and second, the noncovariance of the instantaneous-potential approach to such highly relativistic constituents as u, d, and s quarks.

As an example on the successful side, Le Yaouanc et al. [7] demonstrated the appearance of dynamical (spontaneous) chiral-symmetry breaking by generating the dynamical quark mass as well as the pion as a Goldstone boson in the chiral limit. They studied the power-law interaction ($V \propto r^a$) between massless quarks and antiquarks most exhaustively in the simplest case of harmonic potential ($V \propto r^2$) where the gap equation (Schwinger-Dyson equations) reduced from an integral equation to a differential one. Adler and Davis [8] formulated the renormalization procedure for the Coulomb-like potential and performed a concrete numerical calculation for the linear confining quark-antiquark potential. In the latter case, only infrared divergences appeared. Trzupek [9] applied their renormalization procedure to the more realistic case of the funnel (linear-plus-Coulomb) potential where ultraviolet (UV) divergences were present. This situation was further complicated if finite quark masses were present [11, 12]. Nevertheless, the aforementioned weaknesses led to unsatisfactory quantitative results. For example, the investigations of [7–9] could not yield the value for the pion decay constant better than four to five times smaller than the experimental one, when the meson spectrum was fitted correctly. This was assumed to be the consequence of noncovariance [7]. (Furthermore, it was expected that finite current quark masses would improve the results). Our attention was thus attracted by the covariant generalization of the instantaneous-potential approach to bound states, formulated by Pervushin and collaborators [13–16] and most recently used by Kalinovsky and Weiss [6] in studying the static properties of heavy-light mesons. This approach was first applied to the harmonic potential as the simplest case [17]; however, the first concrete and correct numerical results were obtained by our group [18]. Besides covariance, the effect of the finite current quark masses was also included [18], while previous investigations in this line of research [7–9] had been concerned with massless quarks only. Therefore, they had not been pertinent for studying the dependence of the pion mass on the model parameters since in the chiral limit the pion mass is vanishing. The mass of the pion from our ref. [18] behaves as the square root of the current quark mass, which is precisely the correct behavior of the (pseudo)
Goldstone boson (see, e.g., [19]). The pion decay constant, however, was found in [18] to be $F_\pi \approx 35$ MeV for the harmonic-potential strength which reproduced the experimental pion mass. The results turned out to be practically the same as in [7], $F_\pi = \sqrt{\frac{2}3} f_\pi = \sqrt{\frac{2}3} 20$ MeV $\approx 28$ MeV. The covariant approach removed certain ambiguities present in the definition of the pion decay constant in [7], but did not solve the problem of its too small value. Obviously, further studies of the form of the interaction potential were needed. In the present work we therefore examine the funnel (Coulomb-plus-linear) potential

$$V(r) = V_c(r) + V_L(r) = \frac{4}{3} \left( \frac{z}{r} + \sigma r \right). \tag{1}$$

It has been considered as a prototype of an interquark potential in many papers, among which [6, 8, 9, 11, 30] are just some of the examples. Namely, it is known that, in QCD, the short-distance interactions are dominated by the Coulomb interaction, while in the long-distance (small momentum, or $k \to 0$) regime, times gluon propagator seems to behave as $1/k^4$, corresponding to a linear confining potential in the coordinate language. In other words, although additional terms in the potential such as $O(p^3)$ from the gluon condensate, or an intermediate-range interaction [35] stemming, for example, from instanton effects [36, 37] may well improve the qualitative results, already the funnel potential incorporates the two main qualitative features of the interquark forces arising from QCD: perturbative one-gluon exchange at short distances and linear confinement at long distances, which has made it such a popular choice. However, the Coulomb-type term poses some new difficulties. In cases such as those considered in [6, 8, 9, 11, 30, 31], it causes ultraviolet (UV) divergences in Schwinger-Dyson integral equations, which requires renormalization. Nevertheless, the renormalization of the bound-state equations for quarkonium with instantaneous interaction is still an unresolved and intriguing issue, since the standard procedures used so far either contain momentum-dependent renormalization constants or induce new infra-red (IR) singularities. The study of this issue is in fact the main point of the present work. After sketching in Sect. II how the representation of mesons by bilocal fields leads to the Schwinger-Dyson equation (SDE) with the funnel potential, in Sect. III we formulate a renormalization scheme for the SDE with the funnel potential. We discuss the limitations which such a scheme must suffer when various approximations are introduced. We also compare our renormalization procedure with the ones used so far in this context. In Sect. IV the Salpeter equation for the pion is solved, and in Sect. V the pion decay constant is obtained. In Sect. VI we calculate the $\pi^0 \rightarrow \gamma\gamma$ decay width and conclude in Sect. VII.

## II Mesons as bilocal fields

When trying to model nonperturbative QCD, one may consider a very general interaction kernel $K$ entering in the effective action:

$$W_{\text{eff}} = \int d^4x \left\{ \bar{q}(x) \left[ (i\gamma - \hat{m}) - L(x) \right] q(x) \right. + \left. \frac{1}{2} \int d^4y \bar{q}_2(y) q_1(x) \left[ K(x - y) \right]_{\alpha,\beta_1;\alpha_1,\beta_2} \bar{q}_1(x) q_2(y) \right\}. \tag{2}$$

where $\hat{m}$ is the current quark mass matrix, $\hat{m} = \text{diag}(m_u, m_d, m_s)$. $\alpha$ and $\beta$ are spinor indices, whereas color indices and flavor indices are suppressed. In (2) the summation over repeated indices is understood. We assume that the interaction kernel $K(x - y)$ can lead to a bound $q\bar{q}$ system. In (2) we have introduced $L(x)$, an external operator coupled to the quark current. For example, it can be the leptonic current $l_\mu(x)$:

$$L(x) = \frac{G_F}{\sqrt{2}} l_\mu \gamma^\mu \frac{1}{2} \gamma_5, \tag{3}$$

or a photon $A_\mu(x)$:

$$L(x) = e A_\mu(x) \gamma_\mu = e A(x). \tag{4}$$

Such external operators will make possible the weak and radiative decays, but the internal structure of hadronic bound states will be dictated by the model kernel $K$.

One can construct a theory of meson bound states by eliminating bilinear structures $q_\alpha(x)\bar{q}_\beta(y)$ in favor of a bilocal fields $\chi_{\alpha,\beta}(x, y)$ [20–24], (introduced through the path integral in the generating functional) and then integrating out the remaining quark fields. In this way the action (2) becomes [22]

$$W_{\text{eff}}[\bar{q}, \hat{q}] \rightarrow W_{\text{eff}}[\chi]$$

$$= i \mathcal{N}_c \text{Tr} \left[ (i\gamma - \hat{m} - L - \chi) \right] + \frac{N_c}{2} \left( \chi, K^{-1} \chi \right)$$

$$= \mathcal{N}_c \int \text{Tr} \left[ (i\gamma - \hat{m}) \right] - i \text{Tr} \sum_{n = 1}^\infty \frac{1}{n} \left( \chi, K^{-1} \chi \right)$$

$$\times \left[ (i\gamma - \hat{m})^{-1} (L + \chi) \right]^n + \frac{1}{2} \left( \chi, K^{-1} \chi \right), \tag{5}$$

where we have suppressed all indices and used shorthand:

$$(\chi, K^{-1} \chi) = \int d^4x \int d^4y \chi_{\beta_1,\alpha_1}(x, y) K^{-1}_{\alpha,\beta}(x, y) \chi_{\beta_2,\alpha_2}(y, x) \tag{6}$$

and where $\text{Tr}$ (with the capital “T”) also includes the integration. (Below, “tr”, with small “t”, will denote a trace not including integration.) We can drop the external (weak or electromagnetic) operator $L$ while studying the bound-state equations determining the internal hadron structure. We shall reinstate $L$ later, while studying weak and electromagnetic decays.

We determine the classical solution $\chi_0$ conveniently written as $\chi_0(x - y) \equiv \sum (x, y) - \hat{m} \delta^4(x - y)$, by varying $W_{\text{eff}}$ with respect to $\chi$:

$$\frac{\delta W_{\text{eff}}[\chi]}{\delta \chi} = 0. \tag{7}$$