The estimate from above follows from Proposition 2.1.

Theorem 1.3 is proved.

Theorem 1.4 is derived from Theorem 1.1 in an entirely similar manner.

Corollary 1.5 is obtained from Theorems 1.3 and 1.4 if one takes into account Propositions 4.3, 4.6, 4.7 and Lemma 5.1.

LITERATURE CITED


SUMMATION OF MULTIPLICATIVE FUNCTIONS

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One sharpens the asymptotic formula of Wirsing's theorem in the case \( \tau = 0 \) (E. Wirsing, Math. Ann., 143, 75-102 (1961)).

1. Fundamental Notations. \( P \) is a prime number, \( m, n, \kappa \) are natural numbers, \( \text{f}_1(n), \text{f}_k(n) \) are multiplicative functions, \( \gamma \) is the Euler constant, \( \Gamma(\tau) \) is Euler's gamma function, \( m(x) = \sum_{n \leq x} \frac{\text{f}(n)}{n} \), \( M(x) = \sum_{n \leq x} \text{f}(n), L(x) = \sum_{n \leq x} \text{f}(n) \log n, \prod_{x < p} (1 + \frac{1}{p}) \), \( m_\kappa(x), M_\kappa(x), \prod_\kappa(x) \) are the same functions constructed for \( \text{f}_k(n) \) and \( \Lambda_\text{f}(n) \) is defined by the relation

\[
\text{f}_1(n) \log n = \sum_{d \mid n} \text{f}(d) \Lambda_\text{f}_1 \left( \frac{n}{d} \right),
\]

and \( c_0, c_1, \ldots \) are absolute constants.

2. In [3]* (see also [2, p. 268]) Wirsing proves the following theorem.

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Wirsing's Theorem. If \( f(n) \) is a nonegative multiplicative function satisfying the conditions \( \int (p^k) \leq C_k \) with \( c_2 < k \) and \( k=2,3,\ldots \) for \( x \to \infty \) with a constant \( \gamma > 0 \) we have

\[
\sum_{p \leq x} f(p) = (\gamma + o(1)) \frac{x}{\log x},
\]

then for \( x \to \infty \) we have

\[
M(x) = \left( e^{\gamma x} \right) + o(1) \frac{x}{\log x} \prod(x).
\]

In particular, for \( \gamma = 0 \) we have

\[
M(x) = 0 \left( \frac{x}{\log x} \right) \prod(x).
\]

In \[4\] Wirsing derives the asymptotics (2) for \( x \to \infty \) from the weaker condition

\[
\sum_{p \leq x} \frac{f(p) \log p}{p} = (\gamma + o(1)) \log x.
\]

In \[1\] Levin and Fainleib show on an example that for \( \gamma = 0 \) from (4) one cannot obtain (3).

In the present paper we sharpen asymptotics (3) under condition (1) with \( \gamma = 0 \) for some sufficiently general information relative to the quantity \( o(1) \) which occurs in formula (1).

3. Let \( \gamma(x) \) be a real function satisfying the following conditions:

1. \( \gamma(x) \geq 0 \) and is continuous for \( x \geq 1 \),

2. \( \gamma(x) \to 0 \) for \( x \to \infty \),

3. if \( 0 < y < 1 \), \( x \leq y \leq \infty \), then \( \gamma(x) \sim \gamma(y) \) uniformly relative to \( y \) when \( x \to \infty \),

4. \( \gamma(x) = \int \frac{\gamma(u)}{u} du < \infty \) \( \log x \) , where \( \theta > 0 \) is a constant.

**THEOREM 1.** If \( \gamma(n) \geq 0 \), if there exists \( \lambda > 0 \), \( 0 < \lambda < \frac{1}{k} \), such that

\[
f(p^k) = O(p^{\lambda k}), \quad k \geq 1
\]

and for \( x \to \infty \) we have

\[
\sum_{p \leq x} f(p) \sim \gamma(x) \frac{x}{\log x},
\]

then for \( x \to \infty \) we have

\[
M(x) \sim \frac{\gamma(x)x}{\log x} \prod(x).
\]

**THEOREM 2.** If \( \gamma(n) \geq 0 \) satisfies condition (5) and for \( x \to \infty \) with the constants \( \alpha > 0 \) and \( \beta > 0 \) we have

\[
\sum_{p \leq x} f(p) \sim \frac{\alpha x}{\log x (\log \log x)^\beta},
\]

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