One formulates a conjecture which contains Lehmer's conjecture as a special case and one derives an identity which is equivalent to the negation of the conjecture.

1. Introduction

Lehmer's well-known conjecture consists in the assumption that Ramanujan's function \( \tau(n) \), defined by the expansion

\[
\sum_{n=1}^{\infty} \frac{\tau(n)q^n}{n} = 0, \quad |q| < 1,
\]

does not vanish for \( n \geq 1 \).

D. Lehmer has formulated his conjecture based on numerical computations, shown in [1], that \( \tau(n) \neq 0 \) for several of the first values of \( n \) (viz., for \( n \leq 214,928,639 \)). Relatively recently, J.-P. Serre has given an incomparably deeper basis for this conjecture, showing [2] that the set of the prime numbers \( p \geq 2 \) for which \( \tau(p) \) can be equal to zero has zero density in the set of all prime numbers.

The purpose of this paper consists in the formulation of a conjecture which contains Lehmer's conjecture as a special case and in the proof of an identity which is equivalent to the negation of the mentioned conjecture.

At the second part of the paper I try to show that this identity (Lemma 5 in Sec. 5) is indeed contradictory; if one can succeed to do this, then this will mean the validity of Lehmer's conjecture.

I dedicate with pleasure the present paper to A. I. Vinogradov who had a definite influence on my mathematical taste.

In the same time I would like to express my sincere gratitude to A. I. Vinogradov, L. D. Faddeev, A. B. Venkov and L. A. Takhtadzhyan for their unfailing kind interest in the present paper and for valuable discussions.

2. Fundamental Notations and the Formulation of the Extended Lehmer Conjecture

Let \( G = \text{PSL}(2, \mathbb{Z}) \) be the modular group of fractional-linear transformations of the upper semiplane \( \mathcal{H} \) of the complex variable \( z \). The image of the point \( z \) under the action of the transformation \( g \in G \) will be denoted by \( g \cdot z \),

\[
q \cdot z = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.
\]
For the transformation, given by a matrix with lower row \((c, d)\), we set
\[ j(q, z) = cz + d. \]  

Let \(k\) be an even integer, \(k \geq 4\). A function \(f(z)\) regular on \(\mathcal{H}\) will be called a parabolic form of weight \(k\) if for any \(q \in G\) the function \(f\) satisfies the modular equation
\[ f(qz) = j^k(q, z) f(z) \]  
and if the \(G\)-invariant function \(\frac{j^k}{y^k} |f(z)|^k\), \(y = \Im z\), is bounded on \(\mathcal{H}\).

We denote the space of parabolic forms of weight \(k\) by \(M_k\). It is well known [3] that for even integers \(k \geq 4\), the space \(M_k\) is finite-dimensional and
\[ \text{dim } M_k = \begin{cases} \left[ \frac{k}{12} \right], & k \neq 0 \pmod{12} \\ \left[ \frac{k}{12} \right] - 1, & k = 0 \pmod{12} \end{cases} \]  

This space is generated by Poincaré series of weight \(k\), which here are defined by the equality
\[ P_n(z; K) = \frac{(4\pi n)^{-k/2}}{\Gamma(k/2)} \sum_{g \in G \backslash G} j^{-k}(g, z) e^{2\pi i n g z}, \]  
where \(G_{\infty}\) is the cyclic group generated by the elements \(z \mapsto z + 1\).

For \(k = 4, 6, 8, 10, 12\) and for any integer \(n \geq 1\) the Poincaré series \(P_n(z; k)\) is identically equal to zero since for these values of \(k\) \(M_k\) is empty.

In the case \(\text{dim } M_k \geq 1\) the first Poincaré series \(P_n(z; k)\) with \(1 \leq n < \text{dim } M_k\) form a basis in \(M_k\) and therefore automatically they are not identically zero. The question whether a Poincaré series \(P_n(z; k)\) can be identically equal to zero in the case \(n > \text{dim } M_k \geq 1\) is still open. In connection with this problem, it seems more natural to make the following

Conjecture 1. If the space \(M_k\) of the parabolic forms with an even integer weight \(k \geq 12\) is not empty, then none of the Poincaré series of weight \(k\) is identically zero.

This conjecture is equivalent to the following hypothesis regarding the Fourier coefficients of the basic parabolic forms in \(M_k\).

Conjecture 2. Let \(f_1, \ldots, f_k\), \(k = \text{dim } M_k\), form a basis in \(M_k\) and let \(a_k(n)\) be the Fourier coefficient of index \(n\) of the form \(f_k\). Then for any \(n \geq 1\) at least one of the numbers \(a_1(n), \ldots, a_k(n)\) is not equal to zero.

The equivalence of these conjectures follows from Petersson's formula regarding the inner product of a Poincaré series and an arbitrary parabolic form in \(M_k\).

The inner product in \(M_k\) is introduced by means of the integral
\[ (f_1, f_2) = \int_{\mathcal{O}} f_1(z) \overline{f_2(z)} y^k \, dz, \]  
where \(\mathcal{O} = G \backslash \mathcal{H}\) is the fundamental domain of the modular group \(G\),
\[ \mathcal{O} = \{ z/x = x + iy, -1/2 < x < 1/2, |y| > 1 \} \]