Electrodynamics of a Resistive Sheet Moving Inside a Wave-Guide

D. Schieber, Haifa

Contents: The interaction between a guided wave and a gliding metal sheet is investigated. The Hertz and Fitzgerald superpotentials are resorted to along with the Lorentz transformation and the overall electromagnetic field is obtained.

1 Introduction

In the present paper, the field pattern due to the interaction between electromagnetic waves and a moving sheet is investigated. An externally controlled balancing force is assumed, such that velocity of the sheet is kept constant, whereby the special theory of relativity is readily and exactly applicable. Although in practice — for the present at least — the propulsion velocity of material bodies is much lower than that of light, the Lorentz transformation is resorted to, on the following considerations: (a) it is much simpler than the Galilean transformation in the context of Maxwell’s equations; (b) the superpotentials involved are invariant under it; (c) the final results hold for any velocity arbitrarily close to that of light.

2 Formulation of Problem and Basic Assumptions

The configuration to be investigated is shown in Fig. 1; we introduce — for the time being — a right-handed Cartesian frame of reference \( x, y, z \) in which the running time \( t \) is recorded. Two "superconducting" sheets, extending to infinity along the \( \pm x \) and \( \pm z \) axes, are separated by a distance \( b \). The central plane \( y = 0 \) is blanketed by an extremely thin, non-magnetic metal sheet of width \( \delta \to 0 \). This middle sheet is also assumed to be superconducting, having an electrical conductivity \( \sigma \to \infty \), but the product \( \sigma \delta \) is taken to tend to a finite limit.

The spaces between the three planes are assigned the properties of vacuum; it is further assumed that the middle sheet moves at constant velocity \( v \) along the \( +x \)-axis. What is the field pattern of a guided electromagnetic wave interacting with it?

The problem is solved for (a) Hertz waves and (b) Fitzgerald waves, in that order.

3 The Hertz Vector Superpotential in a Stationary Frame of Reference

Maxwell’s two curl equations link — in free space — the magnetic field intensity \( \mathbf{H} \) and the electric
field intensity $E$ by means of the relations

$$\nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t}, \tag{1}$$

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}, \tag{2}$$

where $\varepsilon_0$ and $\mu_0$ stand -- respectively -- for the free-space permittivity and the free-space permeability; they are related to the velocity $c$ of a plane wave of light through the expression

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \tag{3}$$

Now -- still in free space -- the electromagnetic field is source-free, i.e.

$$\nabla \cdot \mu_0 H = 0, \tag{4}$$

and

$$\nabla \cdot \varepsilon_0 E = 0, \tag{5}$$

so that due to Eq. (4) we may introduce a magnetic vector potential $V$ from which $H$ is derived:

$$\mu_0 H = \nabla \times V. \tag{6}$$

Substituting Eq. (6) in (2) we see -- resorting to an electric scalar potential $\varphi$ -- that $E$ may be derived from

$$E = -\frac{\partial V}{\partial t} - \nabla \varphi. \tag{7}$$

Combination of Eqs. (1) and (7) leads to

$$\nabla \times (\nabla \times V) = -\frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \varphi, \tag{8}$$

or, using the curl curl expansion in a Cartesian system of coordinates:

$$\nabla (\nabla \cdot V) - \nabla^2 V = -\frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \varphi. \tag{9}$$

The Lorentz condition

$$\nabla \cdot V + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0, \tag{10}$$

yields the magnetic wave equation for $V$:

$$\nabla^2 V = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}. \tag{11}$$

Equation (10) is identically satisfied, if the Hertz vector superpotential $\pi$ is introduced through the relations

$$V = -\frac{1}{c^2} \frac{\partial \pi}{\partial t}, \tag{12}$$

and

$$\varphi = \nabla \cdot \pi. \tag{13}$$

For the problem at hand, it suffices to assume that $\pi$ comprises an $x$ component -- only; this assumption will be borne out by the subsequent results.

In view of Eqs. (11) and (12), the Hertz vector satisfies the differential Equation

$$\nabla^2 \pi = \frac{1}{c^4} \frac{\partial^2 \pi}{\partial t^2}, \tag{14}$$

which reduces to

$$\frac{\partial^2 \pi_x}{\partial x^2} + \frac{\partial^2 \pi_x}{\partial y^2} + \frac{\partial^2 \pi_x}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \pi_x}{\partial t^2}. \tag{15}$$

Now, with Eqs. (7), (12) and (13), the electric field components $E_x$, $E_y$ and $E_z$ are obtained from

$$E_x = \frac{1}{c^2} \frac{\partial \pi_x}{\partial x} - \frac{\partial \pi_x}{\partial t}, \tag{16}$$

$$E_y = -\frac{\partial \pi_x}{\partial y}, \tag{17}$$

$$E_z = -\frac{\partial \pi_x}{\partial z}. \tag{18}$$

Whereas -- on account of Eqs. (6), (12) -- the magnetic field reads

$$\mu_0 H_x = 0, \tag{19}$$

$$\mu_0 H_y = -\frac{1}{c^2} \frac{\partial \pi_x}{\partial y}, \tag{20}$$

$$\mu_0 H_z = \frac{1}{c^2} \frac{\partial \pi_x}{\partial z}. \tag{21}$$

We assume that the superpotential $\pi$ pulsates monochromatically at the circular frequency $\omega$, so that -- with the complex amplitude $\mathfrak{A}$ -- we can write

$$\pi = \mathfrak{A} e^{-i\omega t}, \tag{22}$$

where, as usual $i = \sqrt{-1}$.

In order to apply the potential $\mathfrak{A}$ -- we mentally remove -- at this stage -- the gliding sheet, so that the electromagnetic field is confined between the pair of planes $-\delta/2$ only. Because of the configuration at hand, it is assumed that no changes occur along the $+z$-axis, so that Eq. (15) reduces to the two-dimensional Helmholtz equation

$$\frac{\partial^2 \mathfrak{A}}{\partial x^2} + \frac{\partial^2 \mathfrak{A}}{\partial y^2} + \frac{\omega^2}{c^2} \mathfrak{A} = 0. \tag{23}$$

Resorting to an as yet undetermined (wave-) propagation coefficient $\gamma$, together with an as yet unspecified function $f(y)$, we assume

$$\mathfrak{A} = e^{i\gamma(y)e^\gamma f(y)}, \tag{24}$$

so that Eq. (23) reduces to

$$\frac{\partial^2 f}{\partial y^2} + \frac{\omega^2}{c^2} (1 - \gamma^2) f = 0. \tag{25}$$