Superspace Formulation of $N = 2$ Supergravity

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Abstract. A complete formulation of $N = 2$ supergravity in superspace is given including a superfield lagrangian.

1. Introduction
There exist several off-shell versions of $N = 2$ Poincaré supergravity [1-4], but none which is formulated exclusively in terms of superfields in superspace. Here we give such a formulation in close analogy to the presentation of $N = 1$ supergravity in the book of Wess and Bagger [5]*. The method corresponds to the “almost simple” version of Breitenlohner and Sohnius [4], which has also been discussed by Howe [6]. These and the solution of the Bianchi identities by Grimm [7] are the basic references for this paper.

2. Geometry of Extended Superspace
The differential geometry of extended superspace [8] is very similar to the case $N = 1$ [5]. In $N$ extended superspace ($N > 1$), the coordinates are labelled by indices $M \sim (m, \mu, \nu, M)$, where the Grassmann variables carry an additional index $M = 1, \ldots, N$ for some internal symmetry group. With this modification, a lot of equations can be transferred immediately from simple superspace.

The fundamental geometric objects are the vielbein forms $E^\alpha$ and the connection forms $\phi^\alpha$. Torsion and curvature forms are defined through the structure equations:

\[
T^\alpha = \mathcal{D}E^\alpha
\]
\[
R^\alpha{}_{\beta} = d\phi^\alpha{}_{\beta} + \phi^\alpha{}_{\gamma} \phi^\gamma{}_{\beta}
\]

and satisfy the Bianchi identities

\[
\mathcal{D}T^\alpha = \mathcal{D}E^\alpha R_{\alpha}{}^{\delta} = 0
\]
\[
\mathcal{D}R^\alpha{}_{\beta} = 0.
\]

The commutation relation of two covariant derivatives has the form

\[
[\mathcal{D}^\alpha, \mathcal{D}^\beta] \phi^\gamma = - R_{\alpha}{}^{\delta} \mathcal{D}^\delta \phi^\gamma - T^\gamma{}_{\alpha} \phi^\gamma = 0.
\]

The structure group in $N$ extended superspace is the direct product of the Lorentz group and a subgroup of $\text{U}(N)$. In this paper we choose the Lorentz group only. As a consequence, the Lie algebra valued connection and curvature forms have the following properties. The non-vanishing components are

\[
\phi_{a\beta} = \delta_{a}{}^{\alpha} \phi_{\alpha} = \delta_{\beta}{}^{\alpha} \phi_{\alpha}.
\]

The dynamic variables of supergravity are the vielbein and the connection. Since these superfields contain too many component fields, it is necessary to restrict them through covariant conditions. The only tensors available for that are torsion and curvature, and it has been shown that it suffices to impose constraints on the torsion [9]. This will be done in the next section.

3. Constraints for $N = 2$ Supergravity
There are many ways to find constraints for $N = 2$ superspace. They have been derived in the linearized approximation [10, 8], by constraint analysis [11, 12], and from extended supergravity in ordinary space [13]. We shall not discuss these methods here, but simply state a suitable set of constraints. The “usual” constraints, which appear also in $N = 1$ supergravity,
are
\[ T^{A\hat{b}c}_{a\hat{B}c} = 2i \delta^A_{\hat{a}}(\sigma^\dagger)^{\hat{b}} = 0 \]
\[ T^{A\hat{b}c}_{\hat{a}Bc} = T^{\hat{a}Bc}_{A\hat{b}c} = 0 \]
\[ T^A_{\hat{a}c} = T^A_{\hat{a}c} = 0 \]
\[ T_{ab} = 0. \]

The remaining dim 1/2 components of the torsion are reduced to exactly one spinor field through
\[ T^{A\hat{B}}_{\hat{a}Bc} = T^{\hat{a}Bc}_{A\hat{B}} = 0 \]
\[ T^{(A\hat{B}C)}_{\hat{a}\hat{B}c} = T^{\hat{a}Bc}_{(A\hat{B}C)} = 0 \]
(the brackets denote the totally symmetric part). Now there are two possibilities to proceed. First, one could impose an additional constraint on the torsion in order to guarantee the existence of a superspace 2-form satisfying
\[ \mathcal{D}F = 0. \]

Instead, we postulate the existence of a new 1-form \( A \) corresponding to a vector field, and restrict the field strength \( F = \mathcal{D}A \) through
\[ F^{AB}_{\hat{a}B} = 2\sqrt{2} \epsilon_{a\hat{a}} g^{AB} \]
\[ F^{\hat{a}B}_{AB} = -2\sqrt{2} \epsilon_a \delta^{\hat{a}}_{\hat{a}} g_{AB} \]
\[ F^{A\hat{a}}_{\hat{a}B} = 0. \]

These equations are very similar to the usual Yang–Mills constraints \([14]\). Via the Bianchi identity (7), they will yield an additional constraint on the torsion as well as expressions for the other components of \( F \) in terms of \( T \). The constraints given above reduce the number of independent component fields to one irreducible supergravity multiplet with 40 bosonic and 40 fermionic degrees of freedom. The physical fields are the graviton \( g_{\mu\nu} \), two gravitinos \( \psi_{\alpha\beta} \), and the photon \( A_{\mu} \). They are not restricted by any differential equation in \( x \)-space.

4. Solution of the Bianchi Identities

The usual method to find out the consequences of supergravity constraints is to insert them into the Bianchi identities and to solve these identities by introducing a minimal number of superfields. In our case, it turns out that all components of torsion, curvature, and field strength \( F \) can be expressed in terms of only two superfields \( \rho_{\alpha} \) and \( X_{\dagger\hat{a}} \), the complex conjugate fields, and their covariant derivatives \([12]\). The detailed results are summarized at the end of this section. We start with the first Bianchi identity (2), which reads in the vielbein basis
\[ \mathcal{D} \left( \mathcal{D}_{\alpha\hat{a}} T_{\hat{a}\hat{b}\hat{c}} - R_{\alpha\hat{a}} T_{\hat{a}\hat{b}\hat{c}} + T_{\alpha\hat{a}} T_{\hat{b}\hat{c}} \right) = 0 \]
(\( \mathcal{D} \) denotes the graded cyclic sum). Altogether, this identity contains thirty equations from dim 1/2 up to dim 5/2. The independent dim 1/2 identities subject to the constraints (5) are
\[ (\sigma^\dagger)^{\hat{a}} T^{ABc}_{a\hat{B}c} + (\sigma^\dagger)^{\hat{a}} T^{CAB}_{a\hat{B}c} = 0 \]
\[ (\sigma^\dagger)^{\hat{a}} T^{ABc}_{a\hat{B}c} + (\sigma^\dagger)^{\hat{a}} T^{CAB}_{a\hat{B}c} = 0 \]
\[ T^{A\hat{B}}_{\hat{a}Bc} + T^{\hat{a}Bc}_{A\hat{B}} = 0 \]

The first equation is satisfied iff
\[ T^{ABC}_{a\hat{B}c} = 0, \]
and the second one is solved by
\[ T^{ABC}_{a\hat{B}c} = -\delta^A_{\hat{a}} T^{AC}_{a\hat{B}} \]
\[ T^{ABC}_{a\hat{B}c} = -\delta^B_{\hat{a}} T^{AB}_{c\hat{B}} + \delta^C_{\hat{a}} T^{BC}_{A\hat{B}}. \]

The complex conjugate identities yield
\[ T^{ABC}_{A\hat{B}c} = 0 \]
\[ T^{ABC}_{a\hat{B}c} = 0 \]
\[ T^{ABC}_{a\hat{B}c} = 0 \]
\[ T^{ABC}_{a\hat{B}c} = 0 \]

In terms of the superfields \( T^{AB}_{a\hat{B}c} \) and \( T^{aC}_{A\hat{B}} \), the constraints (6) read
\[ T^{AB}_{a\hat{B}c} = 0 \]
\[ T^{aC}_{A\hat{B}} = 0 \]
\[ T^{aC}_{A\hat{B}} = 0 \]

This means that all low-dimensional components of the torsion can be expressed by a single spinor superfield \( p \). We next look for some information about the torsion components \( T^{aC}_{a\hat{B}} \) and \( T^{aC}_{A\hat{B}} \). They appear in the dim 1 identities
\[ R^{A\hat{B}c}_{a\hat{B}c} - 2i(\sigma^\dagger)^{\hat{a}} T^{A\hat{B}c}_{a\hat{B}c} - 2i(\sigma^\dagger)^{\hat{a}} T^{B\hat{a}c}_{a\hat{B}c} = 0 \]
\[ \delta^C T^{A\hat{B}c}_{a\hat{B}c} = -2i(\sigma^\dagger)^{\hat{a}} T^{A\hat{B}c}_{a\hat{B}c} \]
\[ \delta^C T^{A\hat{B}c}_{a\hat{B}c} = -2i(\sigma^\dagger)^{\hat{a}} T^{A\hat{B}c}_{a\hat{B}c} \]
\[ \delta^C T^{A\hat{B}c}_{a\hat{B}c} = -2i(\sigma^\dagger)^{\hat{a}} T^{A\hat{B}c}_{a\hat{B}c} \]

A consequence of the first equation is
\[ T^{ABC}_{a\hat{B}c} = -\epsilon_{\alpha\beta} T^{ABC}_{a\hat{B}c} = -\epsilon_{\alpha\beta} T^{ABC}_{a\hat{B}c} \]
\[ T^{ABC}_{a\hat{B}c} = -\epsilon_{\alpha\beta} T^{ABC}_{a\hat{B}c} = -\epsilon_{\alpha\beta} T^{ABC}_{a\hat{B}c} \]

Inserting this into the second identity gives
\[ T^{AB}_{a\hat{B}c} = 0, \]
\[ T^{aC}_{A\hat{B}} = 0, \]
\[ T^{aC}_{A\hat{B}} = 0, \]

Altogether, we obtain
\[ T^{A\hat{B}c}_{a\hat{B}c} = 0 \]
\[ T^{A\hat{B}c}_{a\hat{B}c} = 0 \]
\[ T^{A\hat{B}c}_{a\hat{B}c} = 0 \]

As can already be seen from (16), the solution of the remaining identities becomes quite complicated due to the existence of low-dimensional torsion components. Fortunately, however, these components can be...