A Derivation of the $U(1)$ Radial Schwinger Model Action Used in the Analysis of Monopole Induced Fermion Number Violating Processes

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Abstract. Starting from the path integral description of the quantum fluctuations around a 't Hooft-Polyakov monopole, we give a derivation of the radial two-dimensional Schwinger model action used in the discussion of the fermion vacuum around the monopole core.

I. Introduction

It is well known that the Schwinger model i.e. two-dimensional QED [1] has remarkable properties as regards the structure of the fermion vacuum. In particular, the vacuum can be a superposition of states with different fermion numbers. In 1981 Rubakov [2] used the close analogy between the radial reduction of quantum electrodynamics of s-wave fermions and the Schwinger model to deduce that 't Hooft-Polyakov monopoles in grand unified theories would induce protons to decay at strong interaction rates if they came into contact with a monopole. The action used by Rubakov in [2] and Callan in [3] to describe the above effect, represents some idealization of the actual situation as regards the quantum fields outside an $SU(2)$ monopole. We give here a derivation of this action, stating all the essential approximations, involved. The full QFT outside an $SU(5)$ monopole involving, in addition, long range non-Abelian colour fields will not be treated, since this is a formidable task which must take into account the way the monopole breaks the colour group from $SU(3)_c$ to $SU(2)_c \times U(1)_c$, as well as the remarks on this and related issues in [4, 5].

In Sect. II we develop the path integral treatment of the quantum fluctuations outside the core of a massive $SU(2)$ 't Hooft-Polyakov monopole. In Sect. III we ask what happens in the limit where the monopole becomes very heavy and the core radius $r_0 \rightarrow 0$. Finally in Sect. IV we describe what happens to the fermion fields in the above limit.

II. Zero Modes and Constraints in the $SU(2)$ 't Hooft-Polyakov Monopole Quantum System

Before giving the path-integral description of the quantum fluctuations around an $SU(2)$ monopole, we must clearly specify the constraints on the quantum system, due to the presence of the monopole. The monopole is essentially a condensation of heavy quanta in a coherent state, represented by the classical monopole solution. The remaining quantum degrees of freedom do not form a complete set with respect to the usual quantum treatment, because of a set of constraints, associated with the appearance of zero energy solutions to the quadratic part of the Hamiltonian. The latter in turn are associated with the symmetries explicitly broken by the monopole configuration, which is described by a set of collective coordinates. In the treatment of the quantum fluctuations of this system we follow the proposal of Osborn in [6], where one uses the background field gauge introduced by 't Hooft for instantons [7] and Polyakov for monopoles [8]. The latter work also serves as a useful reference to part of the treatment we develop here. We examine the simplest Lagrangian, which admits a 't Hooft-Polyakov monopole, namely,

$$L = \int d^3 x \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2} (D_{\mu} \Phi)^* (D^{\mu} \Phi) + V(\Phi^* \Phi) \right],$$

(2.1)

where $A_{\mu} = A_{\mu}^A T^A$, $\Phi = \Phi^A T^A$, $T^A = \frac{1}{2} \tau^A$, ($\tau^A$ are the Pauli matrices) and

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + A_{\mu} \times A_{\nu},$$

$$D_{\mu} \Phi = \partial_{\mu} \Phi + A_{\mu} \times \Phi.$$
monopole electromagnetic field. A static 't Hooft–Polyakov monopole solution to the above system in the spherically symmetric gauge is described by the now familiar form
\[ A^4_0 = 0, \quad A^i_0 = e^{\frac{X^j}{2}} [1 - e^{-r/r_0}] \]
(2.2)
and
\[ \Phi^4 = \frac{X^4}{r} \langle \Phi \rangle_{VAC} [1 - e^{-r/r_0}], \]
where the exact functional dependence near \( r = r_0 \) (the core radius) has been approximated by an exponential, for simplicity. This will not affect our considerations as we take the limit \( r_0 \to 0 \).

In this article we will mostly use the \( SU(2) \) matrix form of the variable \( A_\mu \) and \( \Phi \) and the above classical solution will be denoted by
\[ A^i_\mu = (A^i_0, A^i_4), \]
(2.3)
where
\[ A^i_4 = T^4 A^i_4 \quad \text{and} \quad A^{cl}_4 = T^4 \Phi^4. \]

We introduce the collective coordinates \( X_i(t) \), \( i = 1, 3 \) and \( X_4(t) = \theta(t) \), which are respectively associated with implicit translational invariance and the \( U(1) \) global invariance of the monopole solution. These collective coordinates can be introduced by making a translation of the above solution followed by a \( U(1) \) gauge rotation \([8]\), i.e.
\[ A^i_\mu (x; O) \to A^i_\mu (x - X(t); O) \]
and
\[ A^{cl}_i (x'; \theta) = g^{-1}_0 (x') \frac{1}{i} D_i (A^{cl}_i) g_0 (x') \quad \text{with} \quad x' = x - X(t) \]
where
\[ D_i = \partial_i + A^{cl}_i, \quad D_4 = \Phi^{cl} \]
(2.4)
and
\[ g_0 (x) = \exp[ - i \theta \Phi^{cl}(x)/\mu] \]
[\( \mu \) is the order parameter \( \langle \Phi \rangle_{VAC} \).

If we define the full QFT by expanding around the above solutions, i.e.
\[ A_\mu = A^{cl}_\mu + V_\mu; \quad A^{cl}_\mu = A^{cl}_\mu (x - X(t); \theta(t)) \]
(2.5)
then on inserting (2.5) in the Lagrangian (2.1) we obtain the following constraints \([remembering we are working in an overall temporal gauge i.e. A_0 = 0, V_0 = 0]\)
\[ P^a = \int d^3 x Tr \{ E_\mu (x) z^a_\mu (x; \theta) \} = \]
\[ \int d^3 x Tr \{ V_\mu (x) z^a_\mu (x; \theta) \} = 0 \]
(2.6a)
\[ D_\mu (A^{cl}_\mu) E_\mu (x) = 0 \]
(2.6b)
\[ D_\mu (A^{cl}_\mu) V_\mu (x) = 0 \]
(2.6c)
where (2.6a) corresponds to a residual gauge fixing condition for the quantum fluctuations in the background field of the monopole. In (2.6) the conjugate momenta to \( X^a(t) \) are defined by \( P^a = \delta L/\delta X^a(t) \) and the conjugate momentum to the field \( V_\mu (x) \) is defined by
\[ E_\mu (x; t) = \left( \frac{\delta L}{\delta A^i_j (x); \delta \Phi} \right), \quad j = 1, 3 \]
\[ = (D_\mu A^i_j, D_\mu \Phi). \]
(2.7)
Finally the functions \( z^a_\mu \) define the zero modes, which are defined below, since their precise description is important in the treatment we are following.

A further notational tool is to use an extended space, in which \( X^4 \) represents the compact coordinate \( \theta(t) \), the covariant derivative \( D_4 = \Phi^{cl} \)
(2.8)
since \( \delta_4 f = 0 \) in most cases we will be considering.

The zero mode eigenfunctions \( z^a_\mu \) are chosen to satisfy the background gauge condition \([8]\) :
\[ D_\mu (A^{cl}_\mu) z_\mu (x; \theta) = 0, \]
(2.9)
where
\[ A^{cl}_\mu = A^{cl}_\mu (x - X(t); \Phi(t)). \]

Initially if one derives the constraints (2.6) from the definition of the conjugate momenta then, instead of \( z^a_\mu \), one would obtain \( \delta A^{cl}_\mu / \delta X_\mu \). However, we can always redefine these functions by a suitable gauge transformation. In this way one arrives at the required zero mode functions
\[ z^a_\mu (x - X(t), \theta(t)) = \frac{\delta A^{cl}_\mu}{\delta X_\mu} + D_\mu (A^{cl}_\mu) B^a, \]
(2.10)
where
\[ A^{cl}_\mu = A^{cl}_\mu (x - X(t); \theta(t)) \]
and
\[ B^a = \begin{cases} A^{cl}_i & a = i = 1, 3 \\ - \Phi^{cl} + \omega & a = 4 \end{cases} \]
The function \( \omega \) is defined by the equation \([6]\)
\[ D_\mu (A^{cl}_\mu) D_\mu (A^{cl}_\mu) \omega = V (\Phi^{cl}). \]
(2.11)
Equation (2.10) can be reexpressed in the form
\[ z^a_\mu = \left( - F_i (A^{cl}_\mu); D_j (A^{cl}_\mu) [\omega + \Phi^{cl}] \right) r = i = 1, 2, 3 \]
\[ \left( - D_j (\Phi^{cl}_a); D_\mu (A^{cl}_\mu) \omega \right) \quad r = 4 \]
(2.12)
where the top line represents the translational modes and the bottom line gauge modes \( (D_\mu \omega = \Phi^{cl} \wedge \omega) \). From (2.6a) and (2.10) we see that the momentum \( P_4 \) is conjugate to the global \( U(1) \) angle \( x_4(t) = \theta(t) \) and thus defines the dyon charge degrees of freedom of the monopole, i.e.
\[ Q = \mu^{-1} P_4 = \mu^{-1} \int d^3 x Tr \{ E_\mu z^a_\mu \}. \]
(2.13)