Nonlinear $\sigma$ Model on Supergroup

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Abstract. The geometry of nonlinear $\sigma$ models on supergroup manifolds is studied in the Riemannian formalism, we will show that the argument of Braaten, Curtright and Zachos can be extended to the parallelism of supergroup manifold.

We investigate geometry of nonlinear $\sigma$ models on supergroup manifolds. It is shown that in the Riemannian formalism [1] we can extend the argument of Braaten, Curtright and Zachos (BCZ) [2] on the parallelization of $\sigma$ model on bosonic group manifolds to the case of supergroup manifolds.

We think that nonlinear $\sigma$ model of supergroup manifold is of interest by itself, and also it is interesting in the context of string model [3, 7].

It is known that bosonic $\sigma$ models in 2-dimensions correspond to bosonic strings in curved space because of the similarity of their actions. Its (2-dimensional) supersymmetric extensions are considered as Neveu-Schwarz-Ramond strings [4, 5]. Recently Henneaux and Mezincescu showed that the Green-Schwarz string [6] can also be considered as a 2-dimensional nonlinear $\sigma$ model with a Wess-Zumino term whose target space is SUSY/(Lorentz group), superspace [7].

Therefore we are interested in studying the nonlinear $\sigma$ model whose target space is general superspace. But in order to construct the $\sigma$ models in superspace, we have two different formalisms corresponding to Riemannian superspace [1] and standard non-Riemannian superspace [8]. In order to avoid confusion, we point out that the former can have a torsion by Wess-Zumino term. The Green-Schwarz string is the latter case because its action does not have the kinetic term which is bilinear in the derivative of the fermion fields. In this paper we restrict ourselves to the former case.

In the Riemannian superspace formalism, the line element can be written as

$$ds^2 = dz^a g_{ab} dz^b$$

(1)

where $z^a = (x^a, \theta^a)$. This corresponds to free kinetic part of the $\sigma$ model. In this paper we assume that $g_{ab}$ has the inverse which is necessary for the Riemannian geometry. For any super group we can parametrize the group as

$$U = \exp (i \theta \cdot S) \exp (i x \cdot T)$$

(2)

where $T$ is bosonic and $S$ is fermionic generators. Cartan’s left invariant one form is constructed as

$$U^{-1} dU = i D x \cdot T + i D \theta \cdot S.$$  

(3)

So we get the free kinetic term of 2-dimensional $\sigma$ models in conformal gauge, using vielbein defined as

$$D z^i = d z^a V^i_a$$

$$\mathcal{L} = \partial_a z^a V^i_a V^j_b \eta_{ij} (-1)^{k+b} \partial_a z^b.$$  

(4)

$\eta_{ij}$ is generally an arbitrary $c$-number matrix which have the inverse. In the case of standard superspace the suffices of $\eta_{ij}$ are only bosonic and therefore the metric does not have inverse [9]. From this kinetic term, equation of motion have covariant derivative in terms of Christoffel symbols which can be written by the derivatives of the metric. To this free kinetic part we can add the Wess-Zumino term [10] which is a closed 3-form in 3-dimensions. As will be shown in the following example, a general closed 3-form has an ambiguity. But according to the method of BCZ, we can construct the Wess-Zumino term uniquely to parallelize this manifold.

For this purpose we use matrix representation of supergroup which is found, for instance, in [11]. The metric of tangent space are given by

$$\eta_{ij} = \text{Str}(\lambda_i \lambda_j)$$

(5)
where \( \lambda = (T, S) \) and \([\lambda_\mu, \lambda_\nu] = i f^\lambda_{\mu\nu} \lambda_\lambda\). And by this metric structure constants, \(f_{ijk} f_{lijk} \eta_{kl}\), are fully super antisymmetric. The Wess-Zumino term is

\[
\mathcal{L}_{WZ}^{(3)} = \frac{1}{2} \eta \text{Str}(U^{-1} dU)^3.
\]  

For this setting we can get the spin connection 1-form and curvature 2-form as

\[
\omega_i^j = \frac{1}{2} D z^k f^{jk}_i (\eta - 1) \\
R_i^j = \frac{1}{k} D z^m D z^j (1 - \eta^2) f^m_{l m} f^l_{ij}
\]  

using Maurer-Cartan equation \(dD z^i = gD z^m f^m_{i m f^i_{j k}}\).

For \( \eta = \pm 1 \), \( R_i^j \) vanishes. For an explicit illustration, we investigate a simple model. We choose the algebra \( U(1/1) \) only for simplicity.

\[
[X, Y] = 0 \\
[X, S_1] = -i S_2 \\
[X, S_2] = i S_1 \\
[Y, S_1] = [Y, S_2] = 0 \\
\{S_1, S_2\} = \{S_2, S_1\} = \frac{1}{2} Y \\
\{S_1, S_2\} = 0.
\]  

Although its bosonic part is \( U(1) \otimes U(1) \), this model has a non-vanishing curvature in terms of superspace geometry. The vierbein is

\[
V_a^i = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -\frac{i}{4} \theta_1 & \cos x & -\sin x \\
0 & \frac{i}{4} \theta_2 & \sin x & \cos x
\end{pmatrix}
\]

and a general \( U(1/1) \) left-invariant 3-form is

\[
ii(\eta D \theta_1 D \theta_4 + \lambda D \theta_1 D \theta_2 + \kappa D \theta_2 D \theta_2) Dx,
\]  

where \( \eta, \lambda, \kappa \) are arbitrary constants. (This 3-form is \( U(1/1) \) left-invariant and exact.) But if we carry out the above mentioned method, namely adopting Eq. (6), then we can get correct constants for parallelization, \( \eta = \kappa = \pm 1, \lambda = 0 \). And we get the Lagrangian

\[
\mathcal{L} = 2 \partial_a x \left( \partial_a y - \frac{i}{4} \left( \partial_1 \partial_1 \theta_1 + \partial_2 \partial_2 \theta_2 \right) + i \partial_a \theta_1 \partial_1 \theta_1 \right) + \frac{i}{2} \eta \epsilon^{\rho \theta} \left( \partial_\rho \theta_1 \partial_\rho \theta_4 + \partial_\rho \theta_2 \partial_\rho \theta_2 \right),
\]  

where \( \eta = \pm 1 \).

This Lagrangian is invariant under \( U(1/1)_L \times U(1/1)_R \) transformation. These transformations are isometry for \( g_{ab} \). Explicitly they are

\[
\delta x = 0, \quad \delta y = -\frac{i}{4} \left( \left( \theta_1 \epsilon_1 + \theta_2 \epsilon_2 \right) - \left( \theta_1 \epsilon_1 + \theta_2 \epsilon_2 \right) \cos x \right) + \left( \theta_1 \epsilon_2 - \theta_2 \epsilon_2 \right) \sin x,
\]

\[
\delta \theta_1 = \epsilon_1 \cos x - \epsilon_2 \sin x,
\]

\[
\delta \theta_2 = \epsilon_2 \cos x + \epsilon_1 \sin x,
\]

where the first is translation in fermionic directions and the second is translation in bosonic ones.

We must notice that the term \( i \epsilon_2 \theta_2 \partial_2 \theta_1 \) in this Lagrangian does not have any counterpart in the standard non-Riemannian superspace. But we need this term for the assumed invertibility of the metric.

In conclusion we have shown the parallelizability of nonlinear \( \sigma \) models on the supergroup manifolds with a Wess-Zumino term in the Riemannian formalism. We expect that the wonderful result of [2], geometrostasis, holds also in this case. Details are now under investigation. We are also interested in the standard non-Riemannian formalism, which corresponds to Green-Schwarz and Heterotic strings which we hope to discuss in a future publication.

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References


*The sign of our \( \omega_i^j \) is different from that of \( \omega_i^j \) defined by Braaten et al. [2], according to our definition of \( Dz^i \), cf. (3)