On an Inverse Method in One Dimension

Praveer Asthana and A.N. Kamal
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta T6G 2J1, Canada

Received 29 December 1982

Abstract. Given a sequence of bound-state energies, the inverse Gel'fand-Levitan theory for the one-dimensional Schrödinger equation enables us to uniquely construct a symmetric reflectionless potential which supports those bound states. We test the reliability of this technique by applying it to some known symmetric potentials (confining and non-confining) in one dimension. We also study the systematics under which the method may be reliable.

I. Introduction

Heavy quarkonia are excellent systems for studying the nature of interquark interaction within the framework of nonrelativistic quantum mechanics. The problem of deducing the interquark potential can be tackled in two ways; direct and inverse. In the direct method a potential with a long range confining part and a short range coulomb-like part dictated by one-gluon exchange is fed into the Schrödinger equation and the parameters of the potential are varied to fit the mass spectrum of the quarkonia. In inverse method the hope is that the knowledge of the mass spectrum of the quarkonia can be used to calculate the corresponding potential. In general the latter method is fraught with ambiguities.

The S-wave problem in three dimensions has a formal mathematical similarity with the one-dimensional scattering problem for a symmetric potential. One would, therefore, expect that one-dimensional scattering formalism and the inversion problem in one dimension would have some relevance to the real world, provided that the information used involved only the S-waves of the three-dimensional problem. In particular, one may hope that the knowledge of the low lying S-states in heavy quarkonium systems may be used to invert the corresponding one-dimensional problem and, thereby, derive the interquark potential unambiguously, if not exactly. This inversion program was first undertaken by Quigg, Rosner and Thacker (henceforth, QRT) in a series of papers [1-5].

It has been demonstrated by Kay and Moses, in a series of papers [6, 7], that the inverse problem can be solved uniquely in one dimension for a reflectionless potential with $N$ bound states. The reconstruction of the potential with $N$ bound states requires $2N$ parameters; $N$ of which are the bound-state energies and the other $N$ are related to the normalizations of the bound-state wavefunctions. For a symmetric potential the $N$ parameters related to the wavefunction normalizations are related to the $N$ energy eigenvalues. Thus the knowledge of the $N$ energy eigenvalues suffices to reconstruct uniquely a symmetric reflectionless potential.

This technique can be used as an approximate method to reconstruct a symmetric scattering potential (a potential with a nonvanishing reflection coefficient) in one dimension from its bound-state energies. QRT [1-5] adapted this technique to locally reconstruct a symmetric confining potential.

When one tries to identify the three-dimensional, $l=0$, problem with the one-dimensional problem with a symmetric potential one finds a certain redundancy of information in the one-dimensional problem. In one-dimensional problem only the odd-parity wave functions vanish at the origin while in the three-dimensional problem the wavefunction, $u_{l=0}(r)$, vanishes at the origin in order that $\psi_{n}(r)$ may be finite. Thus the physical states can only be identified with the odd-parity states in the one-di-
mensional case. Thus, if the energy eigenstates of the one-dimensional problem are labelled by $E_1$, $E_2$, etc., in order of increasing energy, only the odd-parity states, $E_2$, $E_4$, etc., can be identified with the physical states of the quarkonium system.

The missing even-parity eigenenergies appear, however, in the expression for the wavefunctions of the reflectionless potential. The values of the physical wave functions at the origin, $\psi_n(0)$, are related to the leptonic widths of these states through the Van Royen-Weisskopf formula [8]. The physical wavefunctions are, as we remarked in the last paragraph, related to the odd-parity wave functions of the one-dimensional symmetric problem. This connection enabled QRT to evaluate the missing even-parity energy eigenvalues.

Once all the energy eigenvalues (in practice, a finite number $N$) of the one-dimensional problem have thus been identified, a unique local reflectionless approximation to the confining quarkonium potential can be reconstructed.

The above program to reconstruct locally a confining potential from the mass spectrum of charmonium and upsilon was undertaken by QRT [1-5]. For completeness the reader is referred to a series of papers by Grosse and Martin [9-12] on the subject of interquark potentials and theorems related to the nonrelativistic problem.

In Sect. II, we summarise the known results of the Gel'fand-Levitan [13] theory for symmetric reflectionless potentials in one dimension. Details are found in papers of QRT and Kay and Moses. In Sect. III, we apply the formalism to approximately reconstruct several symmetric potential wells in one dimension in order to study the systematics under which the method may be expected to work well. In Sect. IV, we discuss the QRT method for the local reconstruction of symmetric confining potentials in one dimension and study a new example of confining potential. We discuss our results in Sect. V.

II. Summary of the Inverse Problem for a Symmetric Reflectionless Potential in One Dimension

The Schrödinger equation in one dimension is,

$$\frac{d^2}{dx^2} u(x, k) + [k^2 - V(x)] u(x, k) = 0 \quad (2.1)$$

where $u(x, k)$ is the wavefunction, $k^2 = E$ is the energy of the particle and $x$ is defined in the infinite interval, $-\infty < x < \infty$. $V(x)$ is a potential localized in such a way that the scattering problem is well defined.

We work with the scattering boundary condition where a plane wave of unit amplitude is incident from the left and is reflected and transmitted with amplitudes $R(k)$ and $T(k)$ respectively. If $V(x)$ is reflectionless,

$$R(k) = 0 \quad \forall \text{ real } k \quad (2.2)$$

then the Gel'fand-Levitan theory [13] gives (details are to be found in QRT [1-5] and Kay and Moses [6, 7]),

$$V(x) = -2 \frac{d^2}{dx^2} \{ \ln \det (I + \hat{A}) \} \quad (2.3)$$

where $I$ is the $N \times N$ unit matrix and $\hat{A}$ is the symmetric matrix given by

$$A = A_n^{1/2} A_m^{1/2} e^{(\kappa_m + \kappa_n)x} \frac{(\kappa_n + \kappa_m)}{\kappa_n} \quad (2.4)$$

with $-\kappa_i^2 (i = 1, 2, \ldots, N)$ the $N$ bound-state energies supported by $V(x)$. $\kappa_i$ are $N$ positive constants related to the normalizations of the bound-state wavefunctions. The normalized bound-state wavefunctions of this potential are given by [6],

$$\psi_n(x) = \frac{1}{A_n^{1/2} e^{\kappa_n x}} \frac{\det (I + \hat{A})^\nu}{\det (I + \hat{A})} \quad (2.5)$$

where $(I + \hat{A})^\nu$ is the matrix $(I + \hat{A})$ with its $n$th column differentiated once with respect to $x$.

In addition if we require the potential to be symmetric,

$$V(x) = V(-x) \quad (2.6)$$

then the necessary and sufficient condition for this is

$$A_n = \prod_{m=1}^{N} \left| \frac{\kappa_m + \kappa_n}{\kappa_m - \kappa_n} \right| \quad ; \quad m = 1, 2, \ldots, N. \quad (2.7)$$

The reflectionless symmetric potential constructed through this prescription is unique [6]. It has the following additional properties:

(i) $V(x)$ is negative for all $x$.
(ii) $V(x)$ approaches zero exponentially as $x \to \pm \infty$.

To test the method numerically we applied the technique to the following reflectionless potential.

**Pöschl-Teller Potential [14]**

This is a symmetric potential in one dimension,

$$V(x) = -\frac{\lambda(\lambda - 1)}{\cosh^2 x}, \quad \lambda > 1. \quad (2.8)$$