On the Large $N_c$ Limit of the $SU(N_c)$ Colour Quark–Gluon Partition Function

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Abstract. The ideal, $SU(N_c)$ coloured, quark–gluon gas partition function is considered, taking the global colour-singlet condition into account. The colour-singlet condition leads to finite volume corrections to the thermodynamical quantities of the gas. Possible effects of these finite volume corrections on lattice QCD at a finite temperature are discussed. Recent Monte Carlo evaluations of the energy density in $SU(2)$ and $SU(3)$ Yang–Mills lattice gauge theories are consistent with such corrections. In the large $N_c$-limit, a phase transition is exhibited which, formally, has the same origin as a large $N_c$-limit phase transition in the two-dimensional $SU(N_c)$ lattice gauge theory.

1. Introduction

The influence of conserved, internal degrees of freedom on the state of a physical system has been studied in detail in the literature. Isospin conservation can e.g. impose constraints on the thermodynamical description of proton–antiproton annihilation (see e.g. [1]) as well as restrictions on the description of coherent pionization in hadronic collisions [2, 7].

The interesting possibility of having a phase transition from a highly compressed or excited state of hadronic matter to a quark–gluon plasma (for an incomplete list of references, see [3]) enforces a study of an ideal (due to asymptotic freedom) quark–gluon gas. Due to the confinement mechanism of QCD, one must impose the constraint of having a colour singlet on the partition function of the system*. Recently, such a study of internal degrees of freedom and the constraints they may impose on the partition function has been considered [4–6].

In the present paper, we will make use of an over-complete set of quasi-coherent states [7–8], where singlets (or any given representation) automatically can be projected out, in order to compute the partition function of an ideal, massless gas with an internal $SU(N_c)$ symmetry. In the large $N_c$-limit, we will exhibit a phase transition which has the same origin as the third-order Gross–Witten phase transition in the two dimensional lattice $SU(N_c)$ gauge theory [9]. For the gluons we must, of course, consider the adjoint representation of $SU(N_c)$. In the large $N_c$-limit, the steepest descent approximation technique used in [9] for the fundamental representation can, however, straightforwardly be extended to any representation. A first-order phase transition at large $N_c$ for an ideal gluon gas is the result of such an analysis, as will be shown below.

One consequence of our considerations is that the, naive, Stefan–Boltzmann limit of a gluon gas cannot be reached when performing Monte Carlo simulations of lattice QCD due to finite lattice effects. Such a conclusion seems to be in accordance with presently available Monte Carlo evaluations of the energy density in $SU(2)$ and $SU(3)$ Yang–Mills lattice gauge theories [10]. As a matter of fact, by taking the finite volume correction into account the approach at high temperatures, to the Stefan–Boltzmann limit in the Monte Carlo calculations of [10] is substantially improved.

2. The Non-Abelian Ideal Boson Gas

Before we discuss the boson gas with internal, non-Abelian degrees of freedom, let us compute the partition function of a photon gas making use of coherent states (see e.g. [11] for a general review of coherent states). The use of coherent states automatically takes
the Bose-Einstein statistics into account. In terms of coherent states, the partition function, \( z \), becomes

\[ z = \text{Tr} \exp(-\beta H) = \int d\phi \langle \phi | \exp(-\beta H) | \phi \rangle. \] (1)

Here

\[ | \phi \rangle = \prod_k \exp \left( -\frac{f_k^* f_k + a_k^* a_k}{2} \right) | 0 \rangle \] (2)

is a coherent state constructed out of the one-particle state \( f_k \) (the index \( k \) runs over all attainable three-momenta) and

\[ \int d\phi = \prod_k \int \frac{d^2 f_k}{\pi} = \prod_k \left( \frac{1}{\pi} \int d\text{Re} f_k d\text{Im} f_k \right). \] (3)

With the free field hamiltonian

\[ H = \sum_k \omega_k a_k^* a_k, \] (4)

where \( \omega_k = |k| \), we obtain

\[ z = \prod_k \int \frac{d^2 f_k}{\pi} \exp(-|f_k|^2(1 - \exp(-\beta \omega_k))) = \prod_k \frac{1}{1 - \exp(-\beta \omega_k)}, \] (5)

which, of course, is a well known result. The reason for presenting this elementary derivation of (5) is that we can make use of the coherent state representation to project out, say, the singlet representation of a non-Abelian Bose gas \([7-8]\). Let \( f_{k,\sigma} \) denote the one-particle state which transforms according to a real representation \( R \) of the group \( G \). The singlet partition function \( Z_s \) then becomes

\[ Z_s = \int d g \int d g_2 < R(g) f | \exp(-\beta H) | R(g_2) f >. \] (6)

By combining (2), (3) and (6) and making use of the invariance of the group measure, we obtain

\[ Z_s = \frac{1}{2\pi(N_c)!} \int \prod_i d\zeta_i \left( \prod_j |2\sin(\pi \xi_i - \xi_j)| \right) \cdot \delta \left( \sum_i \zeta_i \right) \exp((N_c^2 - 1)B_g \zeta(4)) \cdot \exp \left( -\frac{2\pi^4}{3} B_g \sum_{i > j} \left( |\xi_i - \xi_j|^4 - 2|\xi_i - \xi_j|^3 + |\xi_i - \xi_j|^2 \right) \right). \] (10)

For large \( B_g \), i.e. for large \( T \) and/or \( V \), we can apply a steepest-descent method and we obtain

\[ Z_s \approx G \exp((N_c^2 - 1)B_g \zeta(4)) \cdot B_g^{-(N_c^2 - 1)\frac{2}{3}}, \] (12)

where \( G \) is an irrelevant constant. For \( N_c = 2 \), this reduces to a result recently obtained by Gorenstein et al. \([6]\).

By making use of (12) we now obtain for the energy density

\[ \varepsilon = \frac{T^2}{V} \frac{\partial}{\partial T} \ln Z_s = \varepsilon_{SB} \left( 1 - \frac{45}{2\pi^4} \frac{1}{V} \right). \] (13)

where \( \varepsilon_{SB} \) is the naive Stefan–Boltzmann expression for the energy density, i.e.

\[ \varepsilon_{SB} = (N_c^2 - 1)\frac{\pi^2}{15} T^4. \] (14)

Let us comment on the size of the universal correction factor in (13) to the Stefan–Boltzmann distribution. In Monte Carlo simulations of pure Yang–Mills gauge theories one is, of course, forced to work on a finite lattice. The factor \( V \cdot T^3 \) is then determined by the geometrical size of the lattice only \([10]\) and is equal to \( (N/N_c)^3 \), where \( N(N_c) \) is the space (time) extension of the lattice. For a \( 10^3 \times 3 \) lattice, the correction factor then is \( \varepsilon/\varepsilon_{SB} \approx 0.939 \) and for a \( 8^3 \times 3 \) lattice \( \varepsilon/\varepsilon_{SB} \approx 0.881 \). These fairly large corrections on the eigenvalues of \( g \). We can therefore make use of Weyl's parametrization of the reduced group measure \([12]\) to obtain

\[ Z_s = \frac{1}{(N_c)!} \int \prod_i d\zeta_i \left( \prod_j |2\sin(\pi \xi_i - \xi_j)| \right) \cdot \delta \left( \sum_i \zeta_i \right) \exp((N_c - 1)B_g \zeta(4)) \cdot \exp \left( 2B_g \sum_{i > j} \frac{1}{n} \cos n(\xi_i - \xi_j) \right). \] (10)