THE EARTH–MOON POTENTIAL ENERGY

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Summary: The potential energy of the Earth–Moon system is derived and, thus, also the disturbing potential function, responsible for the lunar precession of the Earth’s axis, with preserving the terms from the non-spherical disturbing body. The gravitational fields of the Earth and Moon are considered in the form of a development in terms of spherical harmonics up to n = 4.

In deriving the precession of the Earth's rotation axis, due to the Moon (Sun), and in general in deriving the components of the moments of the external forces, one initially considers the potential function

\[
V_{\Theta G} = G \int \int_G \frac{d \Theta G \ d \Theta}{r},
\]

which can also be called the potential energy of the Earth–Moon system**); \(d \Theta G\) and \(d \Theta\) are the mass elements of the Earth's and lunar bodies, respectively, \(r\) is the distance between these elements (Fig. 1), and \(G\) is the gravitational constant.

Here, we shall derive (1) for the case when the gravitational field of the disturbing body is not spherically symmetrical and its potential, as regards its fundamental features, is described by a series of spherical harmonics. This ties up with [1], but the solution is more general.

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**) Further, we shall only consider the Moon; the procedure for the Sun is formally identical.
The Earth–Moon Potential Energy

We shall consider a geocentric co-ordinate system \( x, y, z \) and a selenocentric system \( x', y', z' \), which will be assumed fixed with the relevant bodies, \( z \) and \( z' \) being the axes of the ellipsoids of inertia of the bodies, \( xz \) the plane parallel with the plane of the astronomical Greenwich meridian, \( x'z' \) the plane of the initial lunar meridian; the east longitude of the axes \( y \) and \( y' \) is equal to 90°. The spherical co-ordinates in the geocentric system (geocentric radius-vector, geocentric latitude and geocentric longitude) will be denoted by \( \theta, \phi, \lambda \), and in the selenocentric system (selenocentric radius-vector, selenocentric latitude and selenocentric longitude) by \( \theta', \phi', \lambda' \). For outer points, instead of \( \Phi \) and \( \Phi' \) we shall use the geocentric \((\delta)\) and selenocentric \((\delta')\) declination, and instead of the east longitudes the hour angles, related to the initial meridians \( xz \) and \( x'z' \), i.e. \( T = -\lambda \) (Greenwich) and \( T' = -\lambda' \). The mass elements \( dm \) will be denoted by the symbols according to the body they refer to; the geocentre is \( O \) and the selenocentre \( O' \).

It holds true that (Fig. 1)

\[
(2) \quad r_{-1} = \ell_{10}^{-1} \sum_{n=0}^{\infty} \left( \ell_{10}^{'} \right)^n \left( \ell_{10}^{'} \right) \psi_n^{(0)} (\cos \psi'),
\]

where

\[
(3) \quad \psi_n^{(0)} (\cos \psi') = \psi_n^{(0)} (\sin \delta') \psi_n^{(0)} (\sin \phi') + \\
+ \sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} \psi_n^{(k)} (\sin \delta') \psi_n^{(k)} (\sin \phi') \cos k(T' + \lambda').
\]

\( \psi_n^{(0)} \) and \( \psi_n^{(k)} \) are Legendre polynomials of the \( n \)-th degree and \( k \)-th order.

For the integration of (1) over both bodies, it is necessary for the quantities, expressing the position of the mass elements of the Earth, to be expressed in the terrestrial system, and the quantities, expressing the position of the mass elements of the Moon, in the lunar system. First of all \((\Pi'_{\theta} = \ell_{10} / \lambda_{10} \lambda')\)

\[
(4) \quad \ell_{10}^{-1} = \Delta_{10}^{-1} \sum_{n=0}^{\infty} \Pi_{10}^{n} \psi_n^{(0)} (\cos \psi'),
\]

where

\[
(5) \quad \psi_n^{(0)} (\cos \psi) = \psi_n^{(0)} (\sin \phi') \psi_n^{(0)} (\sin \delta_{o'}) + \\
+ \sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} \psi_n^{(k)} (\sin \phi') \psi_n^{(k)} (\sin \delta_{o'}) \cos k(A_{\Theta} + T_{\Theta});
\]

\( \delta_{o'}, T_{o'} \) are the geocentric declination and geocentric hour angles of the mass centre of the Moon; analogously, \( \delta_{o}, T_{o} \) are the selenocentric declination and the selenocentric hour angle of the mass centre of the Earth, and \( \Delta_{10} \) is the distance between the mass centres of the Earth and the Moon.