Accumulation Points of the Lagrange and Markov Spectra

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Abstract

In this note several conditions are developed which together guarantee that an element of the Markov spectrum is an accumulation point of the Lagrange spectrum. Every known limit point of the Lagrange spectrum which is the Markov value of an attained non-completely periodic sequence satisfies these criteria. Examples are given to show that the removal of any one of the conditions can result in an element of the Markov spectrum which is not a limit point of the Lagrange spectrum. New elements of the Markov spectrum satisfying these conditions have been included.

1. Notation and Definitions

For any positive irrational number \( t, t = [t_0; t_1, t_2, \ldots] \) will denote the regular simple continued fraction expansion of \( t \). Thus, \( \{t_1, t_2, \ldots\} \) is an infinite sequence of positive integers. For any positive integer \( n \), the bar in the continued fraction \( [0; t_1, \ldots, t_n] \) will signify that \( \{t_i\} \) is the completely periodic infinite sequence of period \( (t_1, \ldots, t_n) \).

The positive integer \( k \) over the bar in \( [0; t_1, \ldots, t_n] \) indicates that \( \{t_i\} \) is the finite sequence consisting of the tuple \( (t_1, \ldots, t_n) \) \( k \) times.

For each positive integer \( i \), we define

\[
\lambda_i(t) = [t_i; t_{i+1}, t_{i+2}, \ldots] + [0; t_{i-1}, \ldots, t_1].
\]

Then the Lagrange value of \( t \) is \( \lambda(t) = \lim_i \lambda_i(t) \), where the supremum is taken over all positive integers. The set of all Lagrange values is called the Lagrange spectrum.

In this note, \( M \) will always denote a doubly infinite sequence \( \{a_i: i = 0, \pm 1, \pm 2, \ldots\} \) of positive integers. For each integer \( i \), we set

\[
\mu_i(M) = [a_i; a_{i+1}, a_{i+2}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots].
\]
Then the Markov value of the sequence $M$ is $\mu(M) = \sup_i \mu_i(M)$, where the supremum is taken over all integers. The set of all Markov values is called the Markov spectrum.

It is known [14, 3] that the Lagrange spectrum is a proper subset of the Markov spectrum. For a more extensive bibliography on these spectra, the reader is referred to [6].

2. Statement and Discussion of the Main Results

In [1] T. W. Cusick proved that the Markov spectrum is precisely the closure of the set of Markov values of sequences which are eventually periodic on both sides. Hence, any element of the Markov spectrum which is not the Markov value of an eventually periodic sequence is a limit point of the Markov spectrum.

Here we determine a set of conditions which together ensure that the Markov value of an eventually periodic sequence is a limit point of both spectra. In Section 6 we include new classes of elements of the Markov spectrum which satisfy these conditions.

We have excluded completely periodic sequences from our study, although our methods do allow us to formulate some remarks on that case in Section 6. A discussion of completely periodic sequences can be found in [2].

In [8] Hall has shown that it is sufficient to consider sequences for which the Markov value is attained. Therefore, we shall restrict our attention to those sequences which are attained at least once and are eventually periodic on each side. We write

$$\mu(M) = \mu_0(M) = [a_0; a_1, a_2, \ldots] + [0; a_{-1}, a_{-2}, \ldots] =$$

$$= [a_0; a_1, \ldots, a_n, c_1, \ldots, c_s] + [0; a_{-1}, \ldots, a_{-m}, b_1, \ldots, b_l].$$

Before stating our results, we make the following definitions.

**Definition:** Let $S$ be a set of real numbers. Then the real number $x$ is said to be a limit point of $S$ if there exists an infinite sequence of distinct elements of $S$ which converges to $x$. The real number $x$ is called an accumulation point (or condensation point) of $S$ if any open interval containing $x$ contains uncountably many distinct elements of $S$.

**Definition:** Let $s, c_1, \ldots, c_s$ be positive integers. We shall call $(c_1, \ldots, c_s)$ a semi-symmetric tuple if either

$$(c_1, \ldots, c_s) = (c_s, \ldots, c_1)$$  \hspace{1cm} (2)