Reverse Problems for Diffusion Equation

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Contents: The paper deals with the analysis of the reverse problems for diffusion equation depending on determination of the boundary conditions of the first and second kind. The idea of proposed method consists on solving the Volterra integral equations of the first kind by means of Tichonov's regularisation method. Theoretical consideration were illustrated by numerical calculations of reverse problems for the uniform $R, C$ transmission line.

Inverse Randwertprobleme der Diffusion


1 Introduction

Many technical problems, e.g. diffusion of electromagnetic field into conducting region, the heat conduction and the voltage or current distribution in $R, C$ lines are described by the diffusion equation.

The progress in optimization and control methods in distributed parameters systems caused necessity of a general solving of reverse problems for diffusion equation. Such problems consist in searching the boundary and initials conditions, which can realize the required time-space field distribution in the investigated region. The reverse problems are problems of the field synthesis and are improperly posed in the sense of Hadamard. In spite of existing numerous works on this subject [1, 2, 3, 5, 7, 8, 9, 10, 11] they have not been solved until now. In this paper we propose the new method of solving the reverse problems depending on determing the boundary conditions of the first and second kind. Its idea consists in applying the Tichonov's regularisation method for solving the integral Volterra equations of the first kind. Similar method has already been applied for solving the problem of the steady magnetic field synthesis [12].

2 Problem Formulation

Let us consider the semi-infinite homogeneous and isotropic space $x \geq 0$, in which the physical quantity $U(x, t)$ satisfies the diffusion equation of the form

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\beta^2} \frac{\partial U}{\partial t}.$$  \hfill (1)

The following reverse problem will be considered: At the point $x = l$ the time dependence of $U(l, t) = h(t)$ is given. The question is, how to determine the boundary condition of the first kind $z(t)$, which can realize the above mentioned function $h(t)$. The complete formulation of this problem is described by the following equations set

$$U(l, t) = h(t), \quad t \geq 0$$  \hfill (2)

$$U(x, 0) = 0, \quad x \geq 0$$  \hfill (3)

$$U(0, t) = z(t), \quad t \geq 0.$$  \hfill (4)

The solution of this problem can be expressed by the Volterra integral equation of the first kind

$$h(t) = \int_0^t N(t - s) z(s) \, ds,$$  \hfill (5)

where

$$N(t) = \frac{1}{2\beta \sqrt{\pi t^3}} \exp \left( -\frac{t^3}{4\beta^2} \right).$$  \hfill (6)

Using the notation

$$K(t, s) = \begin{cases} N(t - s) & \text{for } 0 \leq s \leq t \\ 0 & \text{for } t < s \leq T, \end{cases}$$  \hfill (7)
where $T$ is the time duration of the process, Eq. (5) can be written as the Fredholm integral equation of the first kind

$$h(t) = \int_0^T K(t, s) z(s) \, ds, \quad 0 \leq t \leq T.$$  \hspace{1cm} (8)

The reverse problem defined by Eq. (8) is the problem of synthesis of the time-varying field and is improperly posed in the sense of Hadamard [4, 5, 11, 12, 13]. It means that the solution of this problem may not exist for each given $h(t)$. Even if it should exist, it might not be unique, and it would not depend continuously on $h(t)$. The above mentioned fact makes correct solution of the problem very difficult because of the occasional error occurrence. Solution of the Eq. (8) is then discontinuous with regard to computational data and small calculation errors may cause big errors in final results.

3 Method of Regularisation

For solving the Eq. (8) the method of regularisation is used. The idea is to obtain approximations to the solution of Eq. (8) by minimizing the functional [6, 11, 12, 13]

$$M = \frac{T}{2} \left( W_{j,0}^2 + W_{j,1}^2 + \frac{2}{k-1} \sum_{m=1}^{k-1} W_{j,m}^2 \right) + \frac{\alpha}{2} \left( z_{j+1} - 2z_j + z_{j-1} \right) = B_j,$$  \hspace{1cm} (16)

where $W(t, r)$ and $B(t)$ are defined by Eqs. (12) and (13), respectively.

Solving the equations set (16) we finally obtain the approximated solution of Eq. (8). Solution of the set (16) was done for different values of regularisation parameter $\alpha$ and as the final solution was regarded the one, which satisfied the Eq. (8) with the best accuracy. The dependence of the solution $z(t)$ of the value of regularisation parameter will be discussed in the next chapter.

4 Examples

To illustrate the previous considerations the reverse problem for diffusion equation will be solved for the uniform $R, C$ transmission line. The voltage $U(x, t)$ in the line satisfies the Eq. (1), in which \( t^2/\beta^2 = RC \). Without of losing the generality it will be assumed, that $R = 1 \Omega/m$ and $C = 1 F/m$. The reverse problem (1--4) will be solved for the following function $h_0(t)$

$$h_0(t) = \begin{cases} 2t \text{ V} & \text{for } 0 \leq t \leq \frac{T}{2} \\ \text{V} & \text{for } \frac{T}{2} \leq t \leq T \end{cases}$$  \hspace{1cm} (17)

Shown above function $h_0(t)$ is physically unrealizable. Such situation very often occurs in practice, because this function is usually given as measurements effect in a real physical object and, because of measurements errors, is unrealizable function. Solving Eq. (16) for the data shown below

$$T = 1 \text{ s}, \quad \Delta t = 0.02 \text{ s}, \quad l = 1 \text{ m}, \quad \alpha_1 = 10^{-3}, \quad \alpha_2 = 10^{-3}, \quad \alpha_3 = 10^{-4}, \quad \alpha_4 = 5 \cdot 10^{-6}$$

we obtain the boundary conditions $z(t)$ illustrated in Fig. 1. The solution of diffusion Eq. (1) for the functions $z(t)$ given in Fig. 1. and comparison with the exact function $h_0(t)$ is shown in Fig. 2. As it can be seen, for great value of $\alpha$ we obtain very stable function $z(t)$, but the solution of diffusion equation (solution of the direct problem) for this function $z(t)$ strongly differs from the function $h_0(t)$ given by Eq. (17). For small value of $\alpha$ the function $z(t)$ is more unstable, but the solution of the direct problem is better — the best is for $\alpha = 5 \cdot 10^{-6}$. For $\alpha < 10^{-6}$ the stability of