EDMONDS POLYTOPES AND WEAKLY HAMILTONIAN GRAPHS

Václav CHVÁTAL
Université de Montréal, Montréal, Canada

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Jack Edmonds developed a new way of looking at extremal combinatorial problems and applied his technique with a great success to the problems of the maximal-weight degree-constrained subgraphs. Professor C. St. J.A. Nash-Williams suggested to use Edmonds’ approach in the context of hamiltonian graphs. In the present paper, we determine a new set of inequalities (the “comb inequalities”) which are satisfied by the characteristic functions of hamiltonian circuits but are not explicit in the straightforward integer programming formulation. A direct application of the linear programming duality theorem then leads to a new necessary condition for the existence of hamiltonian circuits; this condition appears to be stronger than the ones previously known. Relating linear programming to hamiltonian circuits, the present paper can also be seen as a continuation of the work of Dantzig, Fulkerson and Johnson on the traveling salesman problem.

0. Notation

As in [6], we denote the “floor of x” (that is, the greatest integer k with \( k \leq x \)) by \( \lfloor x \rfloor \) and the “ceiling of x” (that is, the smallest integer \( k \) with \( k \geq x \)) by \( \lceil x \rceil \).

If \( V \) is a set, we define \( [V] = \{ A \subset V : |A| = 2 \} \). A graph is an ordered pair \( G = (V, X) \), where \( V \) is a set and \( X \subset [V] \). All the graph-theoretical definitions not given here can be found in [5]. A graph is \( n \)-cyclable if, given any set \( S \subset V \) with \( |S| = n \), there is a cycle passing through all points of \( S \). A graph is \( t \)-tough if, for each set \( S \subset V \), the \( S \)-deleted subgraph \( G \setminus S \) has at most \( \max\{t^{-1}|S|, 1\} \) components (see [1]). If \( T, W \) are arbitrary sets, we define

\[
[T, W] = \{ A : |A| = 2, A \cap T \neq \emptyset, A \cap W \neq \emptyset \}.
\]
For a fixed graph \( G = (V, X) \) and sets \( T, W \subseteq V \), we set \( q(T) = |X \cap [T]| \) and \( q(T, W) = |X \cap [T, W]| \). The subgraph \( (T, X \cap [T]) \) induced by \( T \) will be denoted by \( G(T) \); the number of components of \( G(T) \) will be denoted by \( k(T) \).

If \( V \) is a set, we denote by \( \exp^* V \) the set of all proper nonempty subsets of \( V \). We denote by \( \mathbb{N} \) the set of all nonnegative integers. If \( f \) is a real-valued function defined on \( S \), then we write \( f \cdot T \) rather than \( \Sigma \{f(x): x \in S \cap T\} \).

1. Edmonds polytopes

Let us begin with a set of inequalities

\[
\sum_{i=1}^{n} a(i, j) x(i) \leq b(j) \quad (j = 1, 2, ..., m) \tag{1}
\]

\((a(i, j) \text{ and } b(j) \text{ being real numbers})\) which determine a bounded nonempty subset of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Then the set \( M \) of the lattice points of \( \mathbb{R}^n \) (i.e., the points \( x = (x(1), x(2), ..., x(n)) \), where the \( x(i) \)'s are integers) satisfying (1) is finite. Its convex hull is a polytope which can be characterized by a new set of inequalities

\[
\sum_{i=1}^{n} a^*(i, j) x(i) \leq b^*(j) \quad (j = 1, 2, ..., m^*) \tag{2}
\]

The polytope determined by (1) will be denoted by \( P \), the polytope determined by (2) will be denoted by \( E(P) \).

Next consider the following couple of problems:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{n} c(i) x(i), \quad \text{subject to } x \in M, \tag{3} \\
\text{maximize} & \quad \sum_{i=1}^{n} c(i) x(i), \quad \text{subject to } x \in E(P). \tag{4}
\end{align*}
\]

Since the vertices of \( E(P) \) come from \( M \), and \( M \) is a subset of \( E(P) \), we have

\[
\max_{x \in M} \{\Sigma c(i) x(i)\} = \max_{x \in E(P)} \{\Sigma c(i) x(i)\}.
\]