AN EXTENSION OF THE FRANK AND WOLFE
METHOD OF FEASIBLE DIRECTIONS *

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The Frank and Wolfe method of feasible directions is shown to be a case of the more
general computational approach of inner linearization followed by restriction. An extension is
proposed based on this observation. The extended procedure converges, and under certain con-
ditions the asymptotic convergence rate is geometric. Limited computational experience com-
paring the two procedures is reported.

1. Introduction

Methods of feasible directions form a distinctive class of nonlinear
programming algorithms. A variety of procedures which are distin-
guished by the particular combination of direction finding problem and
step size calculation have been suggested [11]. The Frank and Wolfe
procedure [4] was originally developed for a quadratic programming
problem with linear constraints. It is also applicable to the more general
problem:

\[
\maximize f(x), \quad x \in X
\]

where \( x \) is an \( n \)-dimensional vector, \( f \) is real, single-valued, concave and
differentiable on \( X \), and \( X \) is convex.

The procedure can be briefly described as follows:

\textit{step 0}: Find some feasible solution \( \bar{x} \in X \). Go to step 1.

\textit{step 1}: Solve for \( \bar{x} \in X \) such that \( \nabla f(\bar{x})(\bar{x} - \hat{x}) \) is a maximum. If
\( \nabla f(\bar{x})(\bar{x} - \hat{x}) \leq 0 \), stop. If not, go to step 2.

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step 2: Solve for $t$, $0 \leq t \leq 1$ such that $f(\hat{x} + t(\bar{x} - \hat{x}))$ is a maximum. Set $\hat{x} = \hat{x} + t(\bar{x} - \hat{x})$. Go to step 1.

The Frank and Wolfe procedure can also be shown to be a variant of the more general computational approach of inner linearization followed by restriction. This paper points out the correspondence and shows how the more general approach leads to an extension. Necessary and sufficient conditions for optimality are established by considering optimal multipliers and the extended procedure is shown to converge. The performance of the extended procedure is discussed and computational experience reported.

2. Inner linearization and the strategy of restriction

The convex set $X$ can be approximated as closely as desired by inner linearization over an arbitrarily dense grid of points. Such an approximation is always conservative in that no points outside the convex set are included. If we approximate $X$ over an arbitrarily fine base of grid points $(x_1, x_2, ..., x_N)$ in $X$, (2) below is an arbitrarily good approximation of (1).

\[
\text{Maximize } f(\alpha x) \\
\text{subject to } \sum_j \alpha^j = 1 , \\
\alpha^j \geq 0 \text{ for all } j , \\
\text{where } x = [x^1, x^2, ..., x^N]^T , \quad \alpha = [\alpha^1, \alpha^2, ..., \alpha^N] .
\]

Formulation (2) is only conceptual in nature since the base of grid points need not be specified explicitly. Instead, the strategy of restriction allows grid points which are important to the optimization process to be identified as required. Since the grid need not be defined explicitly, we can stipulate that it contains the solution to (1) as well as any other points we might consider during the optimization process. Hence (1) and (2) are equivalent.

The strategy of restriction (see [5] for a more detailed description) specifies that a subset of the nonnegative variables be constrained to zero at each iteration (temporarily removing them from the problem). Therefore, the restricted version of (2) is: