ANALYTICAL SOLUTION OF A DYNAMIC TRANSACTION FLOW PROBLEM

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We analyze a mathematical programming model of a fractional flow process which consists of $n$ sectors and all possible time-dependent streams of flow between, into and out of the sectors. Assuming specific constraints on flow, least cost policies are determined for control of the system transactions involved over a given finite time horizon and over an infinite horizon. Several applications of the model are presented and discussed.

1. Introduction

We consider a fractional flow process of the following type: a system consists of $n$ divisions or sections (ranks, classes, age groups, regions, industries, etc.) which are defined by state vectors $x(t) \in \mathbb{R}_n^+$, where $t > 0$ is an integer denoting an observation point in discrete time. The half-open interval $[t, t+1)$ is defined to be the period $t$. During each period $t$ the state of the system is altered by a flow of exchanges between the various divisions and by exogenous flow into and out of each division. This paper studies the problem of determining least cost policies for control of all system transactions involved over a specified time horizon.

For $1 \leq i \leq n$, $x_i(t)$ represents the amount in group $i$ at time $t$, i.e., the $i^{th}$ component of $x(t)$. For each $1 \leq i, j \leq n$, $s_{ij}(t) \geq 0$ denotes the flow from group $i$ to $j$ during period $t$, that is, that part of $x_i(t)$ which will be contained in group $j$ at the $(t+1)^{st}$ observation; $s_{ij}(t) > 0$ denotes flow out of group $i$ during period $t$; $s_{0j}(t) \geq 0$ denotes the exogenous flow into group $j$ during period $t$; for convenience, we will sometimes use $x_0(t) = \sum_{j=1}^{n} s_{0j}(t)$ to denote the total exogenous input to system...
groups 1 through \(n\) during period \(t\). The state of the system at time zero is given as \(x(0) = x \in R^n\).

From the definitions above, it follows that

\[
\sum_{k=0}^{n} s_{ki}(t) = x_i(t + 1), \quad \sum_{j=0}^{n} s_{ij}(t) = x_i(t)
\]

for each \(1 \leq i \leq n, t \geq 0\).

The system is constrained by linear inequalities which limit the fractional rate of transfer between groups, the fractional rate of flow out of the system from each group, the fraction of total system input received by each group, and the rate of growth of the overall system size. In equation form, for each \(t\) we have

\[
\alpha_{ij} x_i(t) \leq s_{ij}(t) \leq \beta_{ij} x_i(t)
\]

for \(0 \leq i, j \leq n\), where \(0 \leq \alpha_{ij} \leq \beta_{ij} \leq 1\), \(\alpha_{00} = \beta_{00} = 0\) and \(\sum_{j=0}^{n} \alpha_{ij} \leq 1 \leq \sum_{j=0}^{n} \beta_{ij}\) for all \(i, j\) are restrictions on the parameters \(\alpha_{ij}\) and \(\beta_{ij}\), and \(s_{00}(t) \geq 0\) for all \(t \geq 0\). The overall system size at time \(t\) is given by \(\sum_{i=0}^{n} x_i(t)\), and is constrained by the relation

\[
\sum_{i=1}^{n} x_i(t + 1) = \sum_{i=1}^{n} \theta_i(t) x_i(t) \quad \text{for } t \geq 0,
\]

where \(\theta_i(t)\) is a growth factor which describes the "effect" of the present size of group \(i\) on the overall system size in the next period. Condition (1) is equivalent to

\[
\sum_{j=1}^{n} s_{0j}(t) = \sum_{i=1}^{n} (s_{i0}(t) + (\theta_i(t) - 1) x_i(t)),
\]

which states that inputs equal replaced losses plus adjustments to overall size required by specified values of \(\theta_i(t)\). If \(\theta_i(t) = \theta(t)\) for each \(i\), (1) states that from period \(t\) to \(t + 1\), the system will grow at rate \(\theta(t) - 1\) if \(\theta(t) > 1\), or contract at rate \(1 - \theta(t)\) if \(\theta(t) < 1\). Specifying that distinct values of the \(\theta_i(t)\) are possible allows greater flexibility in the applications of the model, as will be shown in Section 2. The values of the \(\theta_i(t)\) may vary over the range

\[
\gamma_i \leq \theta_i(t) \leq \delta_i,
\]