CONVERGENCE AND RESTART IN BRANCH-AND-BOUND ALGORITHMS FOR GLOBAL OPTIMIZATION. APPLICATION TO CONCAVE MINIMIZATION AND D.C. OPTIMIZATION PROBLEMS

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A general branch-and-bound conceptual scheme for global optimization is presented that includes along with previous branch-and-bound approaches also grid-search techniques. The corresponding convergence theory, as well as the question of restart capability for branch-and-bound algorithms used in decomposition or outer approximation schemes are discussed. As an illustration of this conceptual scheme, a finite branch-and-bound algorithm for concave minimization is described and a convergent branch-and-bound algorithm, based on the previous one, is developed for the minimization of a difference of two convex functions.

Key words: Global optimization, nonconvex programming, branch-and-bound, restart procedure, decomposition, outer approximation, concave minimization, d.c. optimization.

1. Introduction

We consider the global optimization problem (P):

\[ \min \{ f(x) : x \in D \} = \min f(D) \]  (1)

where \( D \subset \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R} \).

We assume throughout this paper that \( \min f(D) \) exists.

Though, for its well known inherent difficulties, a global optimization problem in its general form (P) can be considered as essentially intractable by deterministic methods, some promising approaches to broad classes of special global optimization problems have been proposed in recent years. Most prominent examples are concave minimization (e.g. [29, 5, 11, 14, 4, 21, 1, 26, 19, 13, 32, 35]), convex and concave minimization under additional reverse convex constraints (e.g. [8, 9, 4, 31, 3, 27, 28, 35]), d.c. programming (e.g. [33, 34, 35]), Lipschitzian optimization ([4, 18, 20, 24] and references therein). The main methods proposed were cutting plane
algorithms (e.g. [29, 10, 3, 8, 9, 30, 18, 24]), branch-and-bound approaches (e.g. [6, 11, 12, 14, 25, 1]), grid-search techniques (e.g. [7, 20, 23]) and various combinations with many additional devices, such as convex or concave underestimations, decomposition, etc.

Most widely applicable seem to be branch-and-bound methods which have, as compared to other approaches, the potential advantage of being more easily adaptable to parallel processing. A fairly general branch-and-bound scheme has been given in Horst [14] (see also Tuy, Thieu and Thai [32]). There, it has been shown that convergent branch-and-bound procedures can be constructed in many different ways, thus offering a variety of methods that include combinations with cutting planes and can be adapted to many specific problems by exploiting additional structure.

In this paper, first the branch-and-bound conceptual framework in Horst [14] will be slightly further extended and corresponding convergence conditions will be discussed in full generality. This will allow grid-search techniques to be subsumed quite naturally under the branch-and-bound concept, thus leading in several cases to substantial simplifications for the convergence proofs.

Another question of interest in many applications is the possibility to restart a branch-and-bound algorithm in order to take advantage of the information so far obtained, when handling additional constraints in the context of a decomposition or outer approximation scheme. We shall show that this question can be solved rather satisfactorily by providing a restart branch-and-bound procedure incorporated, e.g. in an outer approximation method. With this restart capability branch-and-bound algorithms become more flexible and applicable to a wider range of situations, where decomposition or outer approximation methods are dictated by the specific structure of the problem.

To illustrate the practical use of the above conceptual framework we devote the second part of the paper to the study within this framework of two typical problems of global optimization. The first problem is that of minimizing a concave function over a polytope - a problem that has risen to play a key role in deterministic approaches to global optimization (see e.g. [34]). For the solution of this problem, a new finite branch-and-bound algorithm with restart capability will be presented which is an improved version of a procedure earlier developed by V.T. Ban in his dissertation [1]. The second problem is that of minimizing a d.c. function (i.e. a difference of two convex functions) over a polytope. As shown in [34] (see also references therein) this problem is of considerable interest both from a theoretical and a practical point of view. In a previous work [33] a solution method was proposed for this problem, based on an outer approximation procedure for concave minimization. In the sequel an alternative algorithm will be developed along the lines of the branch-and-bound conceptual scheme discussed in Section 2.

Further questions of importance for constructing branch-and-bound algorithms that cannot be treated here will be discussed in Horst [15] (cf. the Remark at the end of Section 2.2).