THE DESIGN CENTERING PROBLEM AS A D.C. PROGRAMMING PROBLEM

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The following problem is studied: Given a compact set \( S \) in \( \mathbb{R}^n \) and a Minkowski functional \( p(x) \), find the largest positive number \( r \) for which there exists \( x \in S \) such that the set of all \( y \in \mathbb{R}^n \) satisfying \( p(y-x) \leq r \) is contained in \( S \). It is shown that when \( S \) is the intersection of a closed convex set and several complementary convex sets (sets whose complements are open convex) this "design centering problem" can be reformulated as the minimization of some d.c. function (difference of two convex functions) over \( \mathbb{R}^n \). In the case where, moreover, \( p(x) = (x^TAx)^{1/2} \), with \( A \) being a symmetric positive definite matrix, a solution method is developed which is based on the reduction of the problem to the global minimization of a concave function over a compact convex set.

**Key words:** Design centering problem, d.c. programming, difference of two convex functions, complementary convex sets, reverse convex constraints, global minimization of concave functions, outer approximation algorithm.

1. Introduction

Consider a fabrication process in which the quality of a manufactured element is characterized by an \( n \)-dimensional parameter. In order that the manufactured element be acceptable, this parameter must lie in some "region of acceptability" \( S \subseteq \mathbb{R}^n \). But because of unavoidable fluctuations in the manufacturing process, the actual element values of the parameter will deviate from the nominal value in the design. Suppose that the probability for the deviation \( \|y-x\| \) between the actual value \( y \) of the parameter and the designed value \( x \) to be no greater than \( r \):\
\[ \|y-x\| \leq r \]
is equal to \( \alpha(x, r) \):
\[ P\{\|y-x\| \leq r\} = \alpha(x, r), \]
where \( \alpha(\cdot, r) \) is an increasing function of \( r \). Then, for a given nominal value \( x \), the production yield can be measured by the maximal value \( r = r(x) \) such that
\[ B(x, r) := \{y: \|y-x\| \leq r\} \subseteq S. \]

To maximize the production yield, one should, therefore, choose the nominal value \( \bar{x} \) so that \( r(\bar{x}) = \max \{r(x): x \in S\} \). In other words, the problem is to find
\[ \max_{x, r} \{r: B(x, r) \subseteq S\}. \]
In the literature this problem is often referred to as the Design Centering Problem (see e.g. [13]).

In many cases of interest, the region of acceptability is the intersection of a number of convex and complementary convex sets, i.e.

\[ S = C \cap D_1 \cap \cdots \cap D_m, \tag{1} \]

where \( C \) is a closed convex set with \( \text{int} \ C \neq \emptyset \), and \( D_i = \mathbb{R}^n \setminus C_i \) \((i = 1, \ldots, m)\), \( C_i \) being an open convex set in \( \mathbb{R}^n \). Furthermore, instead of the usual norm \( \|y - x\| \) we may consider \( p(y - x) \), where \( p(\cdot) \) is the Minkowski functional of some compact convex body \( B_0 \) containing 0 in its interior, so that

\[ B(x, r) = \{ y : p(y - x) \leq r \}. \tag{2} \]

Thus, mathematically the Design Centering Problem can be formulated as follows:

(P) Given a set \( S \) of the form (1) and a Minkowski functional \( p(\cdot) \), find

\[ \max_{x, r} \{ r : p(y - x) \leq r \text{ implies } y \in S \}. \tag{3} \]

This is an inherently difficult problem, involving two levels of optimization: for every fixed \( x \in \mathbb{R}^n \) find the maximum \( r(x) \) of all \( r \) satisfying

\[ B(x, r) = \{ y : p(y - x) \leq r \} \subseteq S, \]

then maximize the function \( r(x) \) over all \( x \in \mathbb{R}^n \). Even when \( S \) is convex \((S = C, D_1 = \cdots = D_m = \mathbb{R}^n)\) the problem of finding \( r(x) \), for \( x \) given, is known to be NP-hard (unless \( S \) is a polyhedron, see [5]). For this reason, most of the papers published until now on this subject (e.g. [1, 5, 13, 14, 15]) have been restricted either to some particular aspects of the problem, or to the search for a local rather than a global optimum.

The purpose of the present paper is to suggest a new approach to the Design Centering Problem (P), by showing that the function \( r(x) \) is actually a d.c. function, i.e. a function which can be represented as a difference of two convex functions. Thus Problem (P) is a typical d.c. programming problem (maximization of the d.c. function \( r(x) \)).

In view of recent progresses in global optimization theory, especially in d.c. programming theory (e.g. [9, 11, 12, 7, ...]), this result may be of some interest. Hopefully, it will suggest further investigation leading to practical algorithms for solving (P), at least in some important particular cases.

The paper consists of 5 Sections. After the Introduction we discuss a d.c. reformulation of the problem (Section 2). Then in Section 3 we treat the important case where the norm is given by a symmetric positive definite matrix \( A \): \( p(x) = \sqrt{x^TAx} \). In Section 4, a solution method is developed for the problem, on the basis of a specific d.c. representation of the function \( r^2(x) \). Finally, to illustrate how this method works a numerical example is given in Section 5.