A NOTE ON THE SOLUTION OF BILINEAR PROGRAMMING PROBLEMS BY REDUCTION TO CONCAVE MINIMIZATION

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Received 19 October 1985
Revised manuscript received 17 December 1986

We present a new algorithm for solving the bilinear programming problem by reduction to concave minimization. This algorithm is finite, does not assume the boundedness of the constraint set, and uses an efficient procedure for checking whether a concave function is bounded below on a given halfline. Some preliminary computational experience with a computer code for implementing the algorithm on a microcomputer is also reported.

Key words: Outer-approximation algorithm, bilinear programming problem, concave minimization problem.

1. Introduction

A number of problems in engineering design, decision theory, operations research (e.g. constrained bimatrix games [10], three-dimensional assignment [6], multi-commodity network flow [16], production scheduling [8], rectilinear distance location-allocation [14]) can be described by the following general mathematical formulation, often called the bilinear programming problem:

\[
\begin{align*}
\text{(BP)} & \quad \text{minimize} \quad F(x, y) := c^T x + y^T Q x + d^T y, \\
\text{subject to} \quad & x \in D := \{x \in \mathbb{R}^n : Ax \leq a, x \geq 0\}, \\
& y \in E := \{y \in \mathbb{R}^{n'} : By \leq b, y \geq 0\},
\end{align*}
\]

where \( A \) is an \( m \times n \) matrix, \( B \) an \( m' \times n' \) matrix, \( Q \) an \( n' \times n \) matrix; \( a, b, c \) and \( d \) are \( m, m', n, n' \)-vectors respectively. Without loss of generality, we may assume that the polyhedra \( D \) and \( E \) are nonempty, and \( n \leq n' \).

This problem has been extensively studied in the literature for more than twenty years. The earliest methods for its solution were proposed by Mangasarian [10], Mangasarian and Stone [11], and Altman [2]. Subsequently, a number of other algorithms for solving the problem have been developed, including those of Falk [5], Konno [9], Sokirjanskaja [15], Vaish and Shetty [25], [26], Gallo and Ulkućü [7], Sherali and Shetty [13], and Al-Khayyal and Falk [1].
As shown in our earlier paper [17], the bilinear programming problem (BP) in $\mathbb{R}^n \times \mathbb{R}^n$ can always be converted into a **concave programming problem** in $\mathbb{R}^n$. Namely, setting

$$f(x) = \min \{ F(x, y) : y \in E \}$$

$$= e^T x + \min \{ (d + Qx)^T y : y \in E \}$$

(1; x)

it is easily seen that the problem (BP) is equivalent to the following

(CP) minimize $f(x)$

subject to $x \in D$.

Let $P$ denote the set of all $x$ such that $\min \{ (d + Qx)^T y : y \in E \}$ is finite (i.e. $f(x)$ is finite). It is easily verified that $P$ is a polyhedron in $\mathbb{R}^n$ and that $P = \mathbb{R}^n$ if $E$ is bounded. If the problem (BP) has a finite solution we must have $P \supset D$ and the function $f(x)$ is concave and upper semi-continuous, as the pointwise minimum of a family of affine functions (see e.g. [12]).

This close connection between concave programming and bilinear programming suggests that ideas and methods of concave programming could be used for the study of bilinear programming. In fact, Tuy's method [21] originally developed for solving the concave programming problem has been modified, improved and extended in some form or other to the bilinear programming problem (see e.g. Konno [9], Gallo and Ulkucu [7], Vaish and Shetty [25]). Applications of concave programming methods to bilinear programming problems were also discussed in V.T. Ban [3], Benson [4] and T.V. Thieu [19].

It should be noted that most of the mentioned methods use cutting plane [7, 9, 13, 26] or branch-and-bound techniques [1, 5], and assume the boundedness of the sets $D$ and (or) $E$. When the set $D$, for example, is unbounded and (CP) is to be solved, a matter of concern is how to check whether or not the concave function $f(x)$ (to be minimized over $D$) is bounded below on a given halfline, say $\Gamma = \{ x \in \mathbb{R}^n : 0 \leq \theta < \infty \}$. To deal with this question, one may introduce, as in [3], the parametric linear program

$$\min \{ F(x^0 + \theta v, y) : y \in E \}, \quad 0 \leq \theta < \infty, \quad (2)$$

or, alternatively, one may solve the subproblem:

$$\max \{ \theta : f(x^0 + \theta v) \geq f(x^0), \theta \geq 0 \} \quad (3)$$

which is a linear program, as has been shown in [19]. However, the solution of the parametric linear program (2) is computationally expensive, and in both procedures the subproblem to be solved, (2) or (3), explicitly depends upon $x^0$ which implies that different subproblems have to be solved for different points $x^0$. This does not seem to be efficient. Actually, since it is known that the fact whether $f(x)$ is bounded or unbounded below over a halfline depends only on the direction $v$ of the halfline and not on the origin $x^0$ of the halfline, there should exist some criterion for checking this that depends only on $v$. 