CONCAVE DUALITY: APPLICATION TO PROBLEMS DEALING WITH DIFFERENCE OF FUNCTIONS

M. VOLLE
U.E.R. des Sciences, 123 rue Albert Thomas, 87060 Limoges, France

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In this paper a duality is introduced in a concave sense and its relationship with Toland’s duality is studied along with several formulas dealing with the conjugates of differences of convex functions.

Key words: Marginal function, Fenchel conjugation, star-difference.

Introduction

The marginal function associated with a minimization problem takes a prominent part in classical convex duality [7, 13, 14, 20, . . .] and more precisely in the perturbational approach of this duality: indeed, recall that the dual problem is obtained by computing the second conjugate of the marginal function at the origin. Nothing like this occurs in Toland’s duality [25, 26, 27]. However, as one also uses there the notion of perturbed program, a marginal function is inevitably present.

We observe here that this marginal function enjoys pleasant properties (Proposition 1, Theorems 3, 4) and satisfies some variational relations more or less classical in Toland’s duality which are given here in improved forms (Proposition 5, formulas (13), . . ., (18)). Following Hiriart-Urruty who pointed out the relationship between Toland’s duality and the generalized Pschenichnyi’s formula [12], we give some others properties in this framework (Proposition 6, formulas (4), (6), (7), (8), (10), (21), (22)). Proposition 7 gives another version of the generalized Pschenichnyi’s formula. Geometrically, we emphasize several times the part played by the star-difference of sets [10, 17, 18, 19, . . .].

This marginal function leads us, quite naturally, to define what we call the concave duality, that one may consider as an alternative duality to Toland, Singer, and Auchmuty’s dualities [25, 26, 27, 22, 23, 4]. It is obtained by computing the second concave conjugate of the marginal function associated to a minimization problem at the origin, and consists in minimizing a convex function on the whole space. By this way we reduce the duality gap (Proposition 8) but the dual problem is often less attractive. However, we compute this dual for a large class of problems (Proposition 9) and observe that it has more solutions than Toland’s one (Proposition 10). The extremality relations are quite the same as those obtained by Toland (Proposition 11).
Finally we iterate the concave duality and show that the bidual program value is, once more, less than or equal to the Toland's dual value (Corollary 14); we also give a large class of perturbations for which the concave bidual coincides with the primal problem (Corollary 15).

This work makes the most of the fruitful discussions I like to have with Jean-Paul Penot. In particular the idea that Toland's duality could be interpreted by using the second concave conjugate of the perturbation function was contained in (unpublished) lecture notes of his. Although this observation can be considered as a starting point of the present paper, the idea of using the marginal function in connection with Toland's duality seems to be considered here for the first time.

Preliminaries

We recall some properties of the usual extended reals additions that we need in what follows; their proofs can be found in [16].

Let $a, b$ be two extended reals; we set

$$a + b = +\infty \quad \text{if} \quad a \text{ or } b = +\infty,$$

$$a + b = -\infty \quad \text{if} \quad a \text{ or } b = -\infty,$$

$$a + b = a + (-b),$$

$$a - b = a - (-b).$$

In all the other cases the meaning of $a + b$ and $a + b$ is obvious. Then, $+$ and $\#$ are associative, nondecreasing in both variables and we have the following rules:

$$(a - b) = b - a,$$

$$a + b \geq c \iff a \geq c - b,$$

$$\sup_{i \in I} (a + a_i) = a + \sup_{i \in I} a_i,$$

valid for any set $I$ and any extended reals $a, b, c, a_i (i \in I)$, with $\sup_{i \in I} a_i = -\infty$ if $I = \emptyset$. Using the relation

$$\sup_{i \in I} a_i = -\inf_{i \in I} (-a_i)$$

we also have

$$\inf_{i \in I} (a + a_i) = a + \inf_{i \in I} a_i.$$

In the sequel these properties will be used systematically.

1. A digest of Toland's duality [25, 26, 27]

Let $X$ be a set, $Z, Y$ paired locally convex topological vector spaces with topologies compatible with the pairing and $F : X \times Z \to \mathbb{R}$ a function. The Toland's dual of the