CUTTING-PLANE PROOFS IN POLYNOMIAL SPACE

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Following Chvátal, cutting planes may be viewed as a proof system for establishing that a given system of linear inequalities has no integral solution. We show that such proofs may be carried out in polynomial workspace.

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The integer programming problem is to decide if a given system of linear inequalities has an integral solution. Recent progress on this algorithmic question has involved techniques from the geometry of numbers, in the celebrated paper of Lenstra [20] and in results of Babai [1], Grötschel, Lovász and Schrijver [14] and Kannan [16]. One of the things that is apparent in these results is the importance of the fact that if a polyhedron contains no integral vectors then there must be some direction in which it is not very "wide". This idea has been developed more fully by Kannan and Lovász [17], who obtained a theorem which provides much more information on the appearance of such polyhedra. These "width" results have consequences for the construction and analysis of proof systems for verifying that a polyhedron contains no integral vectors. Whereas the integer programming problem is directly related to the question of the equality of $P$ and $NP$, the existence of a polynomial-length proof system for integer programming is equivalent to $NP = co-NP$.

One of the fundamental concepts in the theory of integer programming is that of cutting planes, going back to the work of Dantzig, Fulkerson and Johnson [11] and Gomory [12]. On the practical side, cutting-plane techniques are the basis of very successful algorithms for the solution of large-scale combinatorial and 0–1 programming problems in Crowder, Johnson and Padberg [9], Crowder and Padberg [10], Grötschel, Jünger and Reinelt [13], Padberg, van Roy and Wolsey [21] and elsewhere. On the theoretical side, Chvátal [3, 4, 5, 6] has shown that the notion of cutting planes leads to many nice results and proofs in combinatorics. We will adopt Chvátal's point of view and consider cutting planes as a proof system, in our case for verifying that polyhedra contain no integral vectors.

Perhaps the best known of all proof systems is the resolution method for proving the unsatisfiability of formulas in the propositional calculus. Haken [15] settled a

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long-standing open problem by showing that resolution is nonpolynomial. It is easy to see that proving the unsatisfiability of a formula is a special case of proving that a polyhedron contains no integral vectors, and, using Haken’s result, it can be shown that cutting planes are a strictly more powerful proof system than the resolution system (see [7] for a treatment of this and the relationship of cutting planes and extended resolution).

To define Chvátal’s [5] concept of a cutting-plane proof, consider a system of linear inequalities

$$a_i x \leq b_i \quad (i = 1, \ldots, k).$$

(1)

If we have nonnegative numbers $y_1, \ldots, y_k$ such that $y_1 a_1 + \cdots + y_k a_k$ is integral, then every integral solution of (1) satisfies the inequality

$$\left(y_1 a_1 + \cdots + y_k a_k\right) x \leq y$$

(2)

for any number $y$ which is greater than or equal to $y_1 b_1 + \cdots + y_k b_k$ (the number $y_1 b_1 + \cdots + y_k b_k$ rounded down to the nearest integer). We say that the inequality (2) is derived from (1) using the numbers $y_1, \ldots, y_k$. A cutting-plane proof of the fact that the linear system (1) has no integral solution is a list of inequalities $a_{k+i} x \leq b_{k+i}$ ($i = 1, \ldots, M$), together with nonnegative numbers $y_{ij}$ ($i = 1, \ldots, M$, $j = 1, \ldots, k+i-1$), such that for each $i$ the inequality $a_{k+i} x \leq b_{k+i}$ is derived from the inequalities $a_j x \leq b_j$ ($j = 1, \ldots, k+i-1$) using the numbers $y_{ij}$ ($j = 1, \ldots, k+i-1$) and where the last inequality in the sequence is $0 x \leq -1$. Results of Chvátal [3] and Schrijver [24] imply that a system of rational linear inequalities has no integral solution if and only if this fact has a cutting-plane proof.

The length of a cutting-plane proof is the number, $M$, of derived inequalities. Cook, Coullard, and Turán [7] have shown that results on the ‘width’ of polyhedra imply that if a rational linear system has no integral solution then there exists a cutting-plane proof of this with length bounded above by a function depending only on the number of variables in the system. A consequence of this is that in fixed dimension, the total number of binary digits needed to write down a cutting-plane proof that a rational system $Ax \leq b$ has no integral solution can be bounded above by a polynomial function of the size, in binary notation, of $Ax \leq b$ (see [2, 7]). Unfortunately, the bound on the length of the cutting-plane proofs is necessarily exponential in the number of variables, so for varying dimension we have no guarantee that we can write down our cutting-plane proof in polynomial space. (Again, this is possible if and only if NP = co-NP.) Notice, however, that during the course of a proof it may happen that some of the derived inequalities are no longer needed and so could be removed from our workspace. Thus the amount of space we need in order to carry out a proof may be considerably less than the amount of space it would take to write down the entire list of derived inequalities. So perhaps we can still bound the amount of workspace we need by a polynomial function of the size of $Ax \leq b$. 

[12]

W. Cook / Cutting plane proofs