SOLVING THE LINEAR MATROID PARITY PROBLEM AS A SEQUENCE OF MATROID INTERSECTION PROBLEMS

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In this paper, we present an $O(r^4 n)$ algorithm for the linear matroid parity problem. Our solution technique is to introduce a modest generalization, the non-simple parity problem, and identify an important subclass of non-simple parity problems called 'easy' parity problems which can be solved as matroid intersection problems. We then show how to solve any linear matroid parity problem parametrically as a sequence of 'easy' parity problems.

In contrast to other algorithmic work on this problem, we focus on general structural properties of dual solutions rather than on local primal structures. In a companion paper, we develop these ideas into a duality theory for the parity problem.

Key words: Matroid parity, matroids, polynomial algorithms, matching, matroid intersection.

1. Introduction

Consider an $r \times 2n$ matrix $A$ with columns partitioned into pairs $A(1), A(2), \ldots, A(n)$ called lines. A linearly independent set of lines is a matching and the linear matroid parity problem, hereinafter called the simple parity problem, is to find a matching of maximum cardinality.

The simple parity problem is a common generalization of both the graphic matching problem and the linear matroid intersection problem. It finds applications to scheduling (Fujii et al., 1969; Stallmann, 1982), generalizations of shortest path and spanning tree problems (Lawler, 1976), vehicle routing (Christofides, 1979), experimental design (Lawler, 1976) and network synthesis (Gomory and Hu, 1961; Lawler, 1976; Clements et al., 1986). Lovász (1980a) showed that the problems of finding:

(i) a maximum, circuit-free partial hypergraph of a 3-uniform hypergraph;
(ii) a maximum packing of openly disjoint paths starting and ending in a given set of vertices;
(iii) the minimum number of points which must be 'pinned down' to fix a structure in the plane;
may all be solved as simple parity problems.

In this paper, we present an $O(r^4 n)$ (or $O(r^{3.5} n)$ using fast matrix multiplication) algorithm for the simple parity problem. Our solution technique is to introduce a modest generalization, the non-simple parity problem, and identify an important
subclass of non-simple parity problems called 'easy' parity problems which can be solved as matroid intersection problems. We then show how to solve any linear matroid parity problem parametrically as a sequence of 'easy' parity problems. Analogously, Edmonds' blossom algorithm may be viewed as solving the graphic matching problem as a sequence of 'easy' parity problems. In this case, the 'easy' parity problems are induced bipartite matching problems.

Lawler (1971) posed the parity problem in general matroids. Lovász (1981) and, independently, Korte and Jensen (1982) showed that if we restrict ourselves to oracles to compute rank, this general matroid parity problem admits no polynomially bounded solution procedure. Lovász (1980b) and (1981) developed an $O(n^{10})$ algorithm and a minimax relation for the linear parity problem. Orlin and Vande Vate (1983) developed the algorithm described herein and Stallmann and Gabow (1984) and (1986) developed an $O(r^3n)$ ($O(r^{2.5}n)$ using fast matrix multiplication) augmenting path algorithm.

Although our algorithm is not the most computationally attractive, our solution technique provides considerable insight into the parity problem and may ultimately provide the framework for solving the weighted version. In contrast to other algorithmic work on this problem, we focus on general structural properties of dual solutions rather than on local primal structures. In a companion paper (Vande Vate, 1987), we develop these ideas into a duality theory for the parity problem analogous to the Edmonds–Gallai duality theory for graphic matchings.

In Section 3 we review Lovasz's minimax relation for the simple parity problem. In Section 4, we introduce a modest generalization of the simple parity problem: the non-simple parity problem. We identify an important subclass of non-simple parity problems, called 'easy' parity problems, which, as we show in Section 5, can be solved as matroid intersection problems.

The easy parity problem arises as a relaxation in our algorithm for the simple parity problem. In particular, the easy parity problem arises by partially relaxing the requirement that a matching contain either none or both of the columns of each line. The relaxation is based on a dual solution (or cover) that partitions the lines into subsets. The easy parity problem is to select a set of linearly independent columns containing an even number of columns from each subset of lines. This relaxation extends the role of shrinking in Edmonds' blossom algorithm to the parity problem.

In Section 6, we extend the graphic matching notion of hypomatchability and introduce 'strong covers'. Strong covers ensure that the maximum cardinality of a matching in the easy parity problem is the maximum cardinality of a matching in the original, simple parity problem. In Section 7, we introduce sufficient conditions for a dual solution to be a strong cover. These conditions lead to a special class of strong covers called 'very strong covers'. We show how to modify our solution procedure for the easy parity problem to construct very strong covers.

The intersection procedure described in Section 5 determines the maximum cardinality of a matching, but does not construct a maximum matching. In Section