The maximum entropy method on the mean: Applications to linear programming and superresolution

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Abstract

In this paper, we give two different results. We propose new methods to solve classical optimization problems in linear programming. We also obtain precise quantitative results for the superresolution phenomenon, as observed earlier by practical searchers on specific algorithms. The common background of our work is the generalized moment problem, which is known to be connected with linear programming and superresolution. We describe the Maximum Entropy Method on the Mean that provides solution to the problem and leads to computational criteria to decide the existence of solutions or not.

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1. Introduction

The generalized moment problem consists in the study of the measures \( \mu \) on a measurable space \( U \) satisfying some moment conditions \( \int_U \Phi_i(x) \, d\mu(x) = c_i \), with respect to given weight functions \( \Phi_1, \ldots, \Phi_k \). The measure \( \mu \) may be restricted to be positive or to be absolutely continuous with respect to a given probability measure \( P \), with given lower and upper bounds on its density. Such problem occurs in many areas of physical sciences. Typically, the weight functions are trigonometric polynomials (the given moments are
Fourier coefficients) or algebraic polynomials (giving power moments), see [8,16,22,24] for examples and further references.

The link between generalized moments problems and various optimization problems has been known for a long time (see for example [18,19], and more recently [10]). Two classical examples are:

(I) The minimization of the $L^\infty$-norm under linear constraints. That is: Minimize $\|f\|_\infty$, $f$ being subject to the constraint

$$\int f(x) \Phi(x) \, d\mu(x) = c.$$  

(II) Extremal value of integrals (linear programming): For a given continuous function $\chi$, minimize the integral $\int_U \chi(x) \, d\mu(x)$, $\mu$ being subject to the moment constraints and the prior additional constraints.

These optimization problems have a lot of applications (see the last chapter of [19]): for example, by duality, (I) gives the evaluation of the $L^1$ best approximation by $\Phi$ linear combinations of a given function, see also [4] and [9]. (II) is concerned with semi-infinite linear programming. There is a huge modern literature on the subject. Initial references might be [10,17].

Another problem, connected with the generalized moment problem but little studied, is the quantification of superresolution phenomena. Such phenomena are known since several years ([16]) and may be described as follows. Generalized moment problems often appear in physical sciences [16,22]. In general, the moments $\int \Phi(x) \, d\mu(x)$ are observed with an additive experimental error $\xi_i$. Usually, only the maximum magnitude of the error is known:

$$\|\xi\|_2 := \sqrt{\xi_1^2 + \cdots + \xi_k^2} \leq \epsilon^2.$$  

Positivity or bounds on the density modelize some prior knowledge on the physical object (modelized by $\mu$) to be reconstructed. Very often, $\Phi := (\Phi_1, \ldots, \Phi_k)$ is the vector of trigonometric functions that leads to the Fourier coefficients. The generalized moment problem may be viewed as an undetermined inverse problem since in general infinitely many measures $\mu$ solve the problem. In different areas of applications ([16,22]) algorithms have been proposed that lead to surprisingly precise solutions for the searched measure $\mu$. They are called by the searchers superresolving algorithms. The positivity constraint was suspected to be the reason of this phenomenon, but very little work has been developed to give theoretical foundations and to quantify this important observation. Our aim here is to contribute to a clarification of the phenomenon and give quantitative results about it in the case of bounded densities. Simultaneous efforts in the same direction are made in [5-13].

We first show that superresolution appears only in the case where $c$ is determinate, that is when there is only one measure $\mu$ satisfying the moment constraint and the additional prior constraints. Then, we want to evaluate the modulus of continuity near this determinate point:

$$\omega(c, \epsilon) := \sup_{\mu_1, \mu_2 \in \mathcal{F}(c, \epsilon)} d(\mu_1, p\mu_2),$$