A new method for a class of linear variational inequalities\(^1\)

Bingsheng He

Department of Mathematics, University of Nanjing, Nanjing, 210008, China

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Abstract

In this paper we introduce a new iterative scheme for the numerical solution of a class of linear variational inequalities. Each iteration of the method consists essentially only of a projection to a closed convex set and two matrix-vector multiplications. Both the method and the convergence proof are very simple.

Keywords: Linear variational inequality; Linear complementarity problem; Projection

1. Introduction

We consider a class of linear variational inequalities

\[(\text{LVI}) \quad u \in \Omega, \quad (v - u)^T (Mu + q) \succeq 0, \quad \text{for all } v \in \Omega,\]

where \(M \in \mathbb{R}^{n \times n}\) is a positive semidefinite matrix (not necessarily symmetric), \(q \in \mathbb{R}^n\) and \(\Omega \subset \mathbb{R}^n\) is a closed convex set. The linear complementarity problem

\[(\text{LCP}) \quad u \succeq 0, \quad (Mu + q) \succeq 0, \quad u^T (Mu + q) = 0\]

is a special (LVI) when \(\Omega = \{ u \in \mathbb{R}^n | u \succeq 0 \}\). Variational inequalities, linear complementarity problems have played a significant role in mathematical programming. These subjects have been studied since the mid 1960's starting with the works of Cottle, Dantzig [3], Lemke [9, 10] and developed by many others. There is already a substantial number of algorithms for the numerical solution of linear complementarity problems and variational inequalities [1, 4–8, 12–15]. Our objective in this paper is to offer a new alternative iterative method for solving problem (1). Let \(\Omega^*\) denote the solution set of (LVI) and \(P_\Omega(\cdot)\) denote

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the projection to $\Omega$. Throughout this paper we assume that $\Omega^* \neq \emptyset$ and the projection to $\Omega$ is simple to carry out (e.g. when $\Omega$ is a general orthant, a box, a sphere, a cylinder or a subspace).

2. The method

It is well known [2], that the linear variational inequality (1) is equivalent to the following linear projection equation

\[(\text{LPE}) \quad u = P_{\Omega}[u - (Mu + q)], \quad (3)\]

i.e., to solve (LVI) is equivalent to finding a zero point of the continuous nonsmooth function

\[e(u) := u - P_{\Omega}[u - (Mu + q)]. \quad (4)\]

We state our algorithm as follows:

**Projection and Contradiction Algorithm (PC Algorithm).**

Given $u^0 \in \mathbb{R}^n$. For $k = 0, 1, \ldots$, if $u^k \not\in \Omega^*$, then

\[u^{k+1} = u^k - \rho(u^k)d(u^k), \quad (5)\]

where

\[d(u^k) = (MT + I)e(u^k) \quad (6)\]

and

\[\rho(u^k) = \frac{\|e(u^k)\|^2}{\|d(u^k)\|^2}. \quad (7)\]

Obviously, each iteration of the method consists essentially of only a projection to $\Omega$ and the computation of $Mu$ and $MTe(u)$. We call it a projection and contraction method because in each iteration a projection has to be carried out and the Euclidean distance of the iterates to the solution set monotonically converges to zero, which will be proved in the next section.

3. Convergence

**Theorem 1.** Let $u^* \in \Omega^*$. Then

\[(u - u^*)^T(I + MT)e(u) \geq \|e(u)\|^2, \quad \text{for all } u \in \mathbb{R}^n. \quad (8)\]

**Proof.** Since $\Omega \subset \mathbb{R}^n$ is a closed convex set and $u^* \in \Omega$, we know by the properties of a projection on a closed convex set [11, Appendix B] that

\[\{u^* - P_{\Omega}(u)\}^T\{v - P_{\Omega}(v)\} \leq 0, \quad \text{for all } v \in \mathbb{R}^n.\]