Constant potential primal–dual algorithms: A framework

Levent Tunçel*

Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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Abstract

We start with a study of the primal–dual affine-scaling algorithms for linear programs. Using ideas from Kojima et al., Mizuno and Nagasawa, and new potential functions we establish a framework for primal–dual algorithms that keep a potential function value fixed. We show that if the potential function used in the algorithm is compatible with a corresponding neighborhood of the central path then the convergence proofs simplify greatly. Our algorithms have the property that all the iterates can be kept in a neighborhood of the central path without using any centering in the search directions.

Keywords: Linear programming; Interior point methods; Primal–dual algorithms; Potential function

1. Introduction

In this paper, we study the primal–dual affine-scaling algorithms for linear programs (see Monteiro, Adler and Resende [8], Kojima et al. [2]). We follow up on a recent algorithm by Mizuno and Nagasawa [6] (which utilizes some ideas from Kojima et al. [2]). They suggest starting from an interior primal–dual solution that lies in a one-sided infinity neighborhood of the central path and determine the next iterate based on the Tanabe–Todd–Ye potential function (Tanabe [10], Todd and Ye [12])

\[
\phi(x, s; \rho) := (\rho + n) \log(x^T s) - \sum_{j=1}^{n} \log(x_j s_j),
\]

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where $\rho$ is a positive constant. Given the affine-scaling direction at the current iterate, Mizuno and Nagasawa propose choosing the next iterate such that the value of this potential function does not increase (rather than finding the point which minimizes it along a given search direction). Here we use the same idea with a potential function proposed by Tunçel [13] for $\rho > 0$:

$$
\psi(x, s; \rho) := (\rho + 1) \log \left( \frac{x^T s}{n} \right) - \log(\min_j \{x_j s_j\}).
$$

We show that one can achieve the same complexity bounds achieved by Mizuno and Nagasawa [6] by keeping the potential function $\psi$ constant. Since the potential function $\psi$ has very nice properties related to the infinity-norm neighborhoods of the central path (see [13]), using the contours of $\psi$ we can find a simple expression relating the decrease in the duality gap to the distance of the next iterate from the boundary of the feasible region. As a result our proofs are shorter and simpler. We then define compatibility of potential functions with neighborhoods and generalize the results we had for $\psi$ and one-sided infinity neighborhoods. Finally, we focus on the two-norm neighborhoods of the central path. Proposing a compatible potential function we show that the corresponding primal–dual affine-scaling algorithm has the same worst case iteration complexity as the ones given in [6], but our algorithm is guaranteed to converge to a strictly complementary solution. We also show that in our framework one can easily guarantee that all the iterates lie in a neighborhood of the central path (again a property not shown for existing polynomial time affine-scaling algorithms).

2. Primal–dual affine scaling algorithm

We consider linear programming problems in the following primal (P) and dual (D) forms

(P) \quad minimize \quad c^T x \\
\quad Ax = b, \\
\quad x \geq 0,

(D) \quad maximize \quad b^T y \\
\quad A^T y + s = c, \\
\quad s \geq 0,

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Without loss of generality, we will assume $A$ has full row rank and that there exist interior solutions for both problems, i.e.,

$$
\mathcal{F}_0 := \{(x, s) > 0: x \in F(P), s \in F(D)\} \neq \emptyset,
$$

where $F(P)$ and $F(D)$ denote the set of feasible solutions for the primal and dual problems respectively. Most of the time we will deal only with $s$ as a dual feasible solution. So, whenever we say $s \in F(D)$, we mean that $s \geq 0$ and there exists a $y \in \mathbb{R}^m$ such that