Stochastic analysis of unsaturated transport in soils with fractal log-conductivity distribution

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Abstract: Within the framework of stochastic theory and the spectral perturbation techniques, three-dimensional dispersion in partially saturated soils with fractal log hydraulic conductivity distribution is analyzed. Our analysis is focused on the impact of fractal dimension of log hydraulic conductivity distribution, local dispersivity, and unsaturated flow parameters, such as the soil pore-size distribution parameter and the moisture distribution parameter, on the spreading behavior of solute plume and the concentration variance. Approximate analytical solutions to the stochastic partial differential equations are derived for the variance of asymptotic solute concentration and asymptotic macrodispersivities.

Key words: Stochastic analysis, unsaturated transport, fractals.

1 Introduction

Representing an aquifer's heterogeneity in a complex, three-dimensional numerical model requires extremely detailed measurements of the three-dimensional spatial distribution of the aquifer's hydraulic properties. This may limit the applicability of the deterministic approach to field-scale groundwater transport problems. To overcome these difficulties, researchers usually use a stochastic approach that treats aquifer properties and flow variables as spatial random fields characterized by a limited number of statistical parameters rather than by well-defined deterministic values. The stochastic approach seeks a rational basis for predicting field-scale mean behavior of flow and transport in heterogeneous porous media instead of predicting deterministically the detailed behavior at the local level. The variance of dependent variable is used to characterize the variability associated with the field-scale mean model.

A common feature of mean-behavior stochastic transport models is the concept of effective dispersion coefficients (macrodispersion coefficients) that characterize the spreading induced by the heterogeneities. The evaluation of macrodispersion coefficients requires the derivation of the pore-water velocity autocorrelation function. This in turn depends on the spatial structure of log-conductivity. Estimation of the velocity autocovariance function involves averaging over a large number of realiza-
tions (an ensemble average). To apply the stochastic theory to transport in geologic formations the ergodic hypothesis is used. The single-realization and ensemble approaches should be conceptually equivalent if the scale of the formation heterogeneity is significantly smaller than the transport domain [Gelhar, 1986]. In other words, the single-realization approach is acceptable in the situation where the characteristic length of plume is large compared to the range of correlation of the log-conductivity field.

In reality, the hydraulic properties of sedimentary environments exhibit scale-dependent variance and long-range correlation. Furthermore, as suggested by some field studies [e.g., Freyberg, 1986; Sudicky, 1986], a stationary log hydraulic conductivity field with a definite correlation length cannot always be identified when the correlation scale is of the same order as that of the domain size. Burrough [1983] and Hewett [1986] suggested that scale-dependent phenomena may be characterized by assuming statistically self-similar random fractals. This prompted us to investigate the statistical behavior of unsaturated solute transport in heterogeneous systems whose properties exhibit long-range correlation.

The fractal representation preserves the invariant statistics under the scaling transformation over a range of length scales [Mandelbrot, 1983]. Hewett [1986] discussed the statistical theories of fluid transport in media in which the heterogeneities are fractal. Wheatcraft and Tyler [1988] developed scaling relationships between fractal travel distance and the scale of observation. Ababou and Gelhar [1990] used a band-pass spectral method to solve the stochastic saturated groundwater flow equations for a finite domain. They concluded that the infinite domain of Fourier space can be manipulated to incorporate finite-size effects by using band-pass spectra, with a low wavenumber cut-off proportional to the inverse size of the domain and with a large wavenumber cut-off related to the measurement spacing.

This paper investigates the asymptotic behavior of solute transport and the variance of concentration field in unsaturated soils with a large correlation scale of log-conductivity for the case where the mean hydraulic gradient is vertical. It is assumed that the spatial structure of log-conductivity may be described by self-similar (fractal) statistics. The spectral methodology is used to analyze three-dimensional unsaturated transport.

2 Stochastic solute transport equation for negligible moisture content fluctuations

The stochastic solute transport equation under unsaturated conditions is derived by averaging the local conservative transport equation,

\[
\frac{\partial (\theta C)}{\partial t} = \frac{\partial}{\partial X_i} \left( D_{ij} \theta \frac{\partial C}{\partial X_j} - q_i C \right) \quad i,j = 1, 2, 3
\]  

(1)

where \( C \) = concentration of the transported solute, \( q_i \) = specific discharge in the \( X \) direction, \( \theta \) = soil moisture content, \( D_{ij} \) = local dispersion coefficient tensor (including hydrodynamic dispersion and molecular diffusion), and \( \theta D_{ij} \) = local bulk dispersion coefficient, over the ensemble of soil property realizations. Assuming quasi steady-state conditions, equation (1) becomes

\[
\frac{\partial (q_i C)}{\partial X_i} = D_{ij} \theta \frac{\partial^2 C}{\partial X_i \partial X_j} + \frac{\partial (\theta D_{ij})}{\partial X_i} \frac{\partial C}{\partial X_j}
\]

(2)