THE TRAVELING SALESMAN PROBLEM ON A GRAPH AND SOME RELATED INTEGER POLYHEDRA

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Given a graph $G = (N, E)$ and a length function $l: E \rightarrow \mathbb{R}$, the Graphical Traveling Salesman Problem is that of finding a minimum length cycle going at least once through each node of $G$. This formulation has advantages over the traditional formulation where each node must be visited exactly once. We give some facet inducing inequalities of the convex hull of the solutions to that problem. In particular, the so-called comb inequalities of Grötschel and Padberg are generalized. Some related integer polyhedra are also investigated. Finally, an efficient algorithm is given when $G$ is a series-parallel graph.

Key words: Traveling Salesman Problem, Integer Polyhedron, Facet, Graph, Series-Parallel Graph, Steiner Tree, Polynomial Algorithm.

1. Introduction

Consider a graph $G = (N, E)$ and a function $l: E \rightarrow \mathbb{R}$ which associates the length $l(e)$ to each edge $e \in E$. The classical Traveling Salesman Problem, denoted by TSP, is that of finding a Hamilton cycle $(N, H)$ of $G$ such that $l(H) = \sum_{e \in H} l(e)$ is minimum. (A Hamilton cycle of $G$ is a cycle going exactly once through each node of $G$.) The Traveling Salesman Problem derives its name from the following interpretation: the nodes of $G$ represent cities that must be visited by a salesman and the edges represent roads or other transportation links connecting the cities. One of the cities is the traveling salesman's hometown from which he starts his tour and to which he must return.

Two difficulties arise in stating the TSP as above. First, the graph $G$ may not be Hamiltonian (i.e., $G$ may not have a Hamilton cycle.) Second, even when $G$ is Hamiltonian, the shortest way to visit all the nodes of $G$ may not be to follow a Hamilton cycle. Instead, it may be shorter to go through some nodes more than once and/or use some edges more than once.

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The traditional way to overcome these difficulties is to transform $G$ into a complete graph $K = (N, F)$ on the same node set. The length function $l: F \to \mathbb{R}$ is defined as follows: for every $e \in F$, $l(e)$ is the length of the shortest path of $G$ joining the endpoints of $e$. Solving the TSP on $K$ instead of $G$ clearly resolves the two difficulties just mentioned. Most of the existing literature on the TSP assumes an underlying complete graph.

However, the transformation of $G$ into $K$ has two drawbacks of its own. In most solution techniques a variable is associated with each edge of the graph. Therefore, the TSP on $K$ requires $(|N| - 1)|N|/2$ variables even when the original graph is sparse—which is often the case in applications (Many applications involve planar graphs or graphs with small thickness). The second drawback is that the original problem on $G$ may be easier to solve than the TSP on a complete graph. For example, Ratliff and Rosenthal [9] present a linear time algorithm for a version of the TSP that arises in the context of order picking in a rectangle warehouse. Their algorithm exploits the structure of the underlying graph $G$. We extend their results in Section 5. Another class of graphs for which the TSP can be solved in linear time is given in [2]. For these reasons we prefer to avoid using the complete graph $K$. We propose a different way to overcome the deficiencies associated with the classical formulation of the TSP. Our approach is to introduce a new version of the TSP which we call the Graphical Traveling Salesman Problem. This formulation has also been used successfully by Fleischmann [4].

A tour of a connected graph $G$ is a cycle going at least once through each node of $G$. (Here a cycle may use the same node or the same edge more than once.) The length of a tour $T = (v_1, e_1, \ldots, v_k, e_k, v_1)$ is $l(T) = \sum_{i=1}^{k-1} l(e_i)$. The Graphical Traveling Salesman Problem, denoted by GTSP, consists in finding a tour of $G$ whose length is minimum. Of course GTSP is NP-hard, since, given a graph $G$, the solution of GTSP with the length function $l(e) = 1$ for all $e \in E$, would show whether $G$ is Hamiltonian, a known NP-complete problem.

A graph is Eulerian if it is connected and each of its nodes is incident with an even number of edges. It is well known and easy to prove that if a graph is Eulerian, then it contains a tour using each edge exactly once [1]. Conversely, given a cycle $T$, the graph $H$ induced by the edges of $T$ duplicated as many times as they are used in $T$, is an Eulerian graph. If $T$ is a tour of $G$, then $H$ spans all the nodes of $G$. In other words, the tours of $G$ correspond to the spanning Eulerian graphs obtained from the graph $G$ by removing some edges and duplicating others.

If an edge of $G$ has a negative length, then one can obtain tours of length as small as wanted by using this edge an indefinite number of times. In other words, there is no finite optimum solution. In the remainder we assume that all edge lengths are nonnegative. With this assumption it can be shown that there is an optimum solution using any edge at most twice. (Let $T$ be some tour of $G$ where some edge $e$ is used three times or more. Consider the edge set obtained by taking the edges of $T$ duplicated as many times as they are used in $T$ and by removing two copies of $e$. The graph induced by this edge set is Eulerian and spanning. So it can be traversed by a tour $T'$ Clearly, $l(T') = l(T) - 2l(e) \leq l(T)$.)