Let $D_n$ be the complete digraph on $n$ nodes, and let $P_{LO}$ denote the convex hull of all incidence vectors of arc sets of linear orderings of the nodes of $D_n$ (i.e. these are exactly the acyclic tournaments of $D_n$). We show that various classes of inequalities define facets of $P_{LO}$, e.g. the 3-dicycle inequalities, the simple $k$-fence inequalities and various Möbius ladder inequalities, and we discuss the use of these inequalities in cutting plane approaches to the triangulation problem of input–output matrices, i.e. the solution of permutation resp. linear ordering problems.

**Key words:** Facets of Polyhedra, Linear Ordering Problem, Triangulation Problem, Permutation Problem.

1. Introduction and notation

This paper is a continuation of our paper Grötschel, Jünger and Reinelt (1985) on the acyclic subgraph polytope. The polytope associated with linear orderings is a face of the acyclic subgraph polytope. Our main objective is to investigate which of the inequalities shown to define facets of the acyclic subgraph polytope in our former paper also define facets of the linear ordering polytope. We adopt the notations in graph theory and polyhedral theory of that paper.

A linear ordering (or permutation) of a finite set $V$ with $|V|=n$ is a bijective mapping $\sigma: \{1, 2, \ldots, n\} \to V$. For $u, v \in V$ we say that $u$ is 'better than' or 'before' $v$ if $\sigma^{-1}(u) \prec \sigma^{-1}(v)$. Among all possible linear orderings of $V$ we want to find a linear ordering which is the best according to some criterion. In many applications a 'value' or a 'cost' can be associated with a linear ordering in the following way. For every two elements $u, v \in V$ a value $c_{uv}$ and a value $c_{vu}$ are given which can be interpreted as the profit we obtain from having $u$ 'before' $v$ resp. $v$ 'before' $u$ in a linear ordering. Then the total value of a linear ordering clearly is given by

$$\sum_{\sigma^{-1}(u) \prec \sigma^{-1}(v)} c_{uv}.$$ 

Given a linear ordering of the nodes $V$ of a digraph then the arc set $\{(u, v) | \sigma^{-1}(u) \prec \sigma^{-1}(v)\}$ forms an acyclic tournament on $V$, and similarly, if $(V, T)$ is an acyclic tournament then this induces a linear ordering of $V$. Using this graph theoretical interpretation we can state the linear ordering problem as follows.
Given a complete digraph \( D_n = (V, A_n) \) with arc weights \( c_{ij} \) for all \((i, j) \in A_n\), find a spanning acyclic tournament \((V, T)\) in \( D_n \) such that

\[
c(T) := \sum_{(i, j) \in T} c_{ij}
\]

is as large as possible. This problem is NP-complete, cf. Garey and Johnson [1979].

For ease of notation, whenever we shall use the word tournament in the sequel we shall mean the arc set of a spanning tournament.

The linear ordering problem is sometimes also called the **permutation problem** (Young (1979)) or the **triangulation problem** (Korte and Oberhofer (1968), (1969)) and is closely related to the feedback arc set problem and the acyclic subgraph problem, see Grötschel, Jünger and Reinelt (1985) for a discussion of these relations, and see Lenstra (1973), Marcotorchino and Michaud (1979) and Wessels (1981) for real world applications of the linear ordering problem in triangulation of input-output matrices, scheduling (minimizing average weighted completion time), sports, archeology, social sciences, and psychology.

In subsequent constructions we will frequently have to manipulate acyclic tournaments. The following notation will be convenient: \( \langle i_1, i_2, \ldots, i_n \rangle \) denotes the arc set of the acyclic tournament \( \{(i_j, i_k) | j < k\} \), i.e. \( \langle i_1, i_2, \ldots, i_n \rangle \) is the acyclic tournament induced by the linear ordering defined by the mapping \( \sigma(j) = i_j \) for \( j = 1, \ldots, n \).

### 2. Dimension, valid inequalities

Let \( D_n = (V, A_n) \) be the complete digraph of order \( n \), and set

\[
A_n := \{A \subseteq A_n | A \text{ is acyclic}\}, \tag{2.1}
\]
\[
\mathcal{T}_n := \{T \subseteq A_n | T \text{ is an acyclic tournament}\}. \tag{2.2}
\]

Clearly, \( \mathcal{T}_n \subseteq A_n \) and for every \( A \in A_n \) there exists a \( T \in \mathcal{T}_n \) with \( A \subseteq T \). Given weights \( c_{ij} \) for every arc \((i, j) \in A_m\), then the acyclic subgraph problem (for \( D_n \)) is to solve \( \max\{c(A) | A \in A_n\} \) while the linear ordering problem can be stated as \( \max\{c(T) | T \in \mathcal{T}_n\} \). In the following way polytopes can be associated with the acyclic subgraph problem and the linear ordering problem.

Let \( \mathbb{R}^m, m := |A_n| = n(n-1) \), denote the real vector space where every component of a vector \( x \in \mathbb{R}^m \) is indexed by an arc \((i, j) \in A_n\). For convenience we write \( x_{ij} \) instead of \( x_{(i,j)} \). For every arc set \( A \subseteq A_n \) the incidence vector \( x^A \in \mathbb{R}^m \) of \( A \) is defined as follows: \( x^A_{ij} = 1, \) if \((i, j) \in A\), and \( x^A_{ij} = 0, \) if \((i, j) \notin A\). The **acyclic subgraph polytope** \( P_{AC}^n \) on \( D_n \) is the convex hull of the incidence vectors of all acyclic arc set in \( D_n \), i.e.

\[
P_{AC}^n = \text{conv}\{x^A \in \mathbb{R}^m | A \in A_n\}. \tag{2.3}
\]

(This polytope is denoted \( P_{AC}(D_n) \) in Grötschel, Jünger and Reinelt (1985). We use here the shorter notation (2.3).) Similarly, the **linear ordering polytope** \( P_{LO}^n \) on